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ELASTO-PLASTIC ANALYSIS OF AXISYMMETRICALLY LOADED
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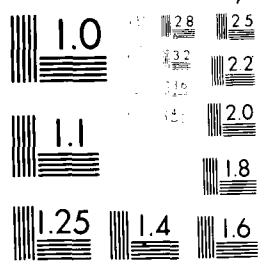
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DEPARTMENT OF OCEAN ENGINEERING

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

CAMBRIDGE, MASSACHUSETTS 02139

ELASTO PLASTIC ANALYSIS OF AXISYMMETRICALLY LOADED
CIRCULAR PLATES USING GREEN'S FUNCTIONS

BY

PETER DUDLEY EARECKSON

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B.S. Mech. Eng., Cornell University
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CIRCULAR PLATES USING GREEN'S FUNCTIONS

by

PETER DUDLEY EARECKSON

Submitted to the Department of Ocean Engineering
in May 1986 in partial fulfillment of the
requirements for the Degrees of Ocean Engineer and
Master of Science in Civil Engineering

ABSTRACT

A boundary integral formulation for the analysis of circular plate bending under lateral loads is developed using Green's functions. The formulation specifically applies to annular plates with arbitrary boundary conditions. The plate bending solution in the plastic range is determined using a numerical method of incremental loading. A computer program to perform the required calculations was developed and is presented. Results for three case studies are included and compared with results obtained by other methods. Plate behavior in the elastic range is in excellent agreement with other analytical solutions, and in the plastic range is in reasonable agreement with published results obtained using a finite element method.

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NOMENCLATURE

a	-	plate outer radius
b	-	plate inner radius
c	-	plate outer radius for Green's function
D	-	flexural rigidity
d	-	plate inner radius for Green's function
E	-	Young's modulus
h	-	plate thickness
I	-	area moment of inertia of beam cross section
K_r	-	radial curvature
K_θ	-	tangential curvature
L	-	beam length
M, M_G	-	elastic moments for beam bending problem
M_r, M_{Gr}	-	elastic radial moments per unit length
M_r^P	-	plastic radial moment per unit length
\bar{M}_r	-	actual radial moment per unit length
$M_\theta, M_{G\theta}$	-	elastic tangential moments per unit length
M_θ^P	-	plastic tangential moment per unit length
Q, Q_G	-	shears for beam bending problem
Q_r, Q_{Gr}	-	elastic radial shears per unit length
\bar{Q}_r	-	actual radial shear per unit length
q, q_G	-	distributed loads per unit area

R_r	-	radial radius of curvature
R_θ	-	tangential radius of curvature
r	-	radial distance from plate center
r_0	-	radial position of ring load
S_y	-	yield stress
w, w_G	-	deflections
x	-	distance along beam
z	-	lateral distance from plate center
$\delta \epsilon_P$	-	plastic strain increment
ϵ_r	-	total radial strain
ϵ_r^e	-	elastic radial strain
ϵ_r^P	-	plastic radial strain
ϵ_θ	-	total tangential strain
ϵ_θ^e	-	elastic tangential strain
ϵ_θ^P	-	plastic tangential strain
ϵ_y	-	yield strain
$\dot{r}, \dot{\theta}_G$	-	slopes
Γ	-	plate boundary
θ	-	angular position in tangential direction
∇^4	-	biharmonic operator
λ, λ_G	-	loads per unit length for beam bending problem
ρ_G	-	magnitude of Green's function ring load
σ_e	-	equivalent stress
σ_r	-	radial stress
σ_θ	-	tangential stress
ν	-	Poisson's ratio
Ω	-	plate domain

CHAPTER 1

INTRODUCTION

The solution of plate bending problems in the plastic range has been approached using finite element, and more recently, boundary integral methods (also called boundary element methods). Unlike the finite element method, which discretizes the inside of the plate into a number of small elements, the boundary integral method discretizes only the plate boundary, thereby reducing the order of the problem by one.

Many different approaches have been used to formulate plate bending problems using boundary integrals, with perhaps the most attractive proposed by Stern^{<1>} and Bezine^{<2>}. These authors, working independently, developed a direct boundary integral method using Green's functions to solve the general plate bending problem in the elastic range. This work has been extended by Moshaiov and Vorus^{<3>} to include plastic behavior.

Symmetry is commonly used to reduce the size of the system

of equations needed to analyze a symmetric structure. In the case of axisymmetrically loaded circular plates the problem becomes one-dimensional. For the linear elastic case, it has a closed form analytic solution. However, some authors have instead treated this circular plate problem as a two-dimensional problem, perhaps for the sake of demonstration (Kamiya and Sawki^{<4>}). They have used symmetry to allow discretization of a portion of the boundary instead of the entire boundary.

This thesis develops a general formulation for solving annular plate bending problems using boundary integrals that reduces the problem to essentially a one-dimensional problem by the use of a "ring" type Green's function. Such a formulation is most useful, inspite of the closed form solution, as it allows the analysis of plates with different boundary conditions and loads with one algorithm. Moreover, it permits a solution in the plastic range, which is not possible with a conventional analytic approach. In addition, the simplicity of the one-dimensional formulation given here has a unique significance for the teaching of boundary element methods.

Butterfield^{<5>} has developed a boundary element formulation for the one-dimensional elastic beam bending problem. Here, a similar approach is taken for the annular plate. It is also extended to include non-linear behavior using an incremental load method as outlined by Moshaiov and

Vorus. Numerical results for three different annular plate configurations are presented and compared to results obtained using other methods. A discussion of these results follows, as well as suggestions for future work in this area.

CHAPTER 2

DEVELOPMENT OF THE GOVERNING DIFFERENTIAL EQUATION

This chapter reviews the derivation of the governing differential equation for a circular plate subjected to a symmetrically distributed lateral load. The derivation is based on that provided by Timoshenko⁶, using Kirchhoff's assumptions. It is, therefore, applicable only to the linear-elastic case. Treatment of non-linear behavior is addressed in Chapter 5.

2.1 Moment-Curvature Relationships

The first step in developing the governing differential equation for a circular plate is to find a relationship between moment and curvature. Curvature in the radial direction (K_r) and the tangential direction (K_θ) for a symmetrically loaded circular plate is found using geometrical considerations. K_r , which is the inverse of the radius of curvature in the radial direction (R_r), can vary as a function of the distance (r) from the plate center. Examining the plate of Figure 2-1 at a small

radial distance dr away from r , the slope (ϕ) can be expressed in terms of the radial distance r and the deflection (w) as follows

$$\phi = - \frac{dw}{dr} \quad (2-1-1)$$

The radial radius of curvature is

$$R_r = \frac{dr}{d\phi}$$

giving an expression for the radial curvature in terms of w

$$K_r = \frac{1}{R_r} = \frac{d\phi}{dr} = - \frac{d^2 w}{dr^2} \quad (2-1-2)$$

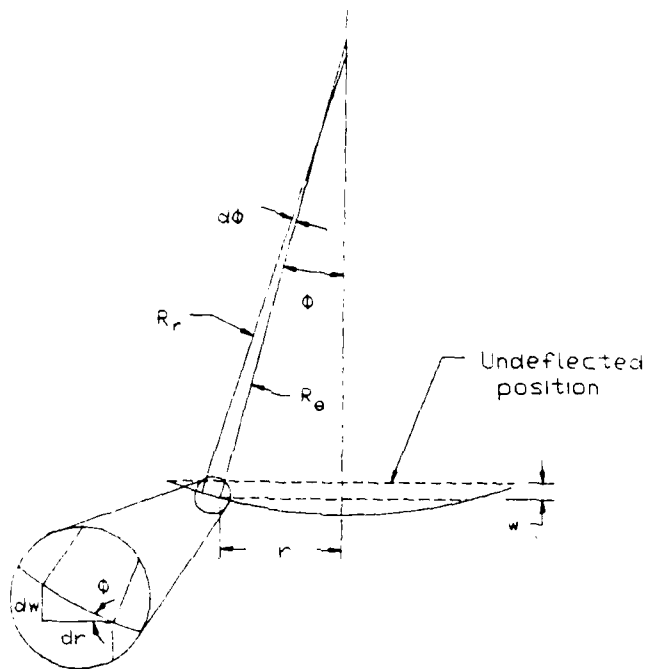


Figure 2-1. Deflection, slope, and curvature relationships in circular plate bending

Due to symmetry, the tangential radius of curvature is constant at any given radius r , and for small deflections can be expressed as

$$R_{\theta} = \frac{r}{\frac{dw}{dr}}$$

giving an expression for the tangential curvature

$$K_{\theta} = \frac{1}{R_{\theta}} = \frac{\dot{\theta}}{r} = -\frac{1}{r} \frac{dw}{dr}$$

To relate moment to curvature, consider the small plate element illustrated in Figure 2-2. Define M_r and M_{θ} , the radial and tangential moments per unit length respectively, as follows:

$$M_r = \int_{-h/2}^{h/2} \sigma_r z \, dz$$

$$M_{\theta} = \int_{-h/2}^{h/2} \sigma_{\theta} z \, dz$$
(2-1-3)

where h is the plate thickness. Assuming the plate is thin and hence experiences a two-dimensional stress state, Hooke's law yields the following expressions relating stresses to strains

$$\sigma_r = \frac{E}{(1-\nu^2)} \left[\epsilon_r^e + \nu \epsilon_{\theta}^e \right]$$

$$\sigma_{\theta} = \frac{E}{(1-\nu^2)} \left[\epsilon_{\theta}^e + \nu \epsilon_r^e \right]$$
(2-1-4)

where E is Young's modulus, ν is Poisson's ratio, and ϵ_r^e and ϵ_{θ}^e are the elastic radial and tangential strains, respectively. In Chapter 5, which discusses non-linear behavior, a distinction will be made between the elastic strain and the total strain. In the plastic range, the total strain includes elastic and plastic components, in accordance with the following expressions:

$$\epsilon_r = \epsilon_r^e + \epsilon_r^p$$

$$\epsilon_\theta = \epsilon_\theta^e + \epsilon_\theta^p$$

where the superscript p refers to the plastic strain components. Since Chapter 2 treats only elastic behavior, the total strain is simply equal to the elastic strain.

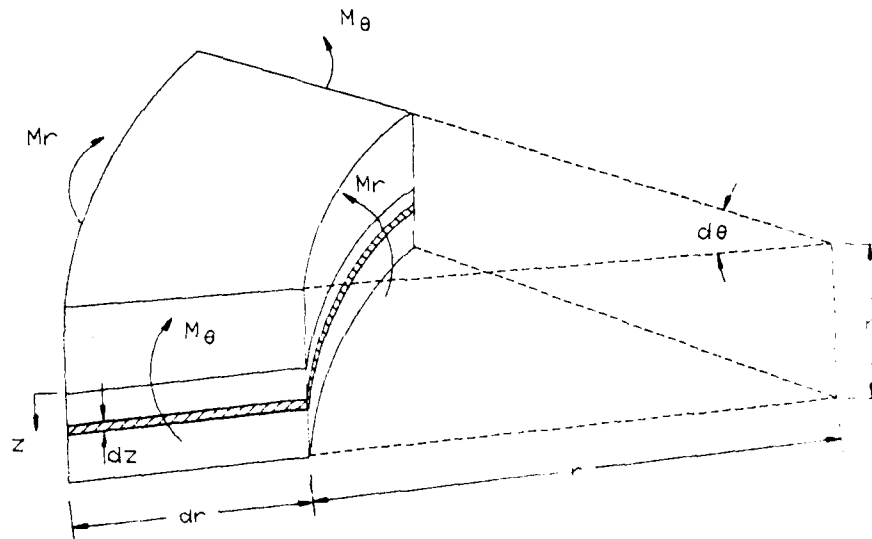


Figure 2-2. Differential circular plate element.

Substituting Eq. (2-1-4) into Eq. (2-1-3) gives

$$M_r = \frac{E}{(1-\nu^2)} \int_{-h/2}^{h/2} \left[\epsilon_r + \nu \epsilon_\theta \right] z \, dz \quad (2-1-5)$$

$$M_\theta = \frac{E}{(1-\nu^2)} \int_{-h/2}^{h/2} \left[\epsilon_\theta + \nu \epsilon_r \right] z \, dz$$

Finally, from geometrical considerations, strain and curvature are related by

$$\epsilon_r = K_r z = - \frac{d^2 w}{dr^2} z \quad (2-1-6)$$

$$\epsilon_\theta = K_\theta z = - \frac{1}{r} \frac{dw}{dr} z$$

which, when substituted into Eq. (2-1-5), give the desired moment-curvature relationship for a circular plate

$$M_r = -D \left[\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right] \quad (2-1-7)$$

$$M_\theta = -D \left[\frac{1}{r} \frac{dw}{dr} + \nu \frac{d^2 w}{dr^2} \right]$$

where D , the flexural rigidity, is given by

$$D = \frac{E h^3}{12 (1-\nu^2)} \quad (2-1-8)$$

2.2 Equilibrium Equations

For the moment-curvature relationships to be useful, an equilibrium equations in terms of moments and shears must be found. The governing differential equation is then developed by substituting the expressions relating moment to curvature into the equilibrium equation.

To find an equilibrium equation, consider first the laterally loaded circular plate element illustrated in Figure 2-3, where Q_r is the radial shearing force per unit length and q is the distributed load (force per unit area). The couples acting along edges ab and cd due to radial bending moments are

$$\text{ab:} \quad M_r r d\theta$$

$$\text{cd:} \quad \left(M_r + \frac{dM_r}{dr} dr \right) (r + dr) d\theta$$

The shear forces acting along the edges are

$$\text{ab:} \quad Q_r r d\theta$$

$$\text{cd:} \quad \left(Q_r + \frac{dQ_r}{dr} dr \right) (r + dr) d\theta$$

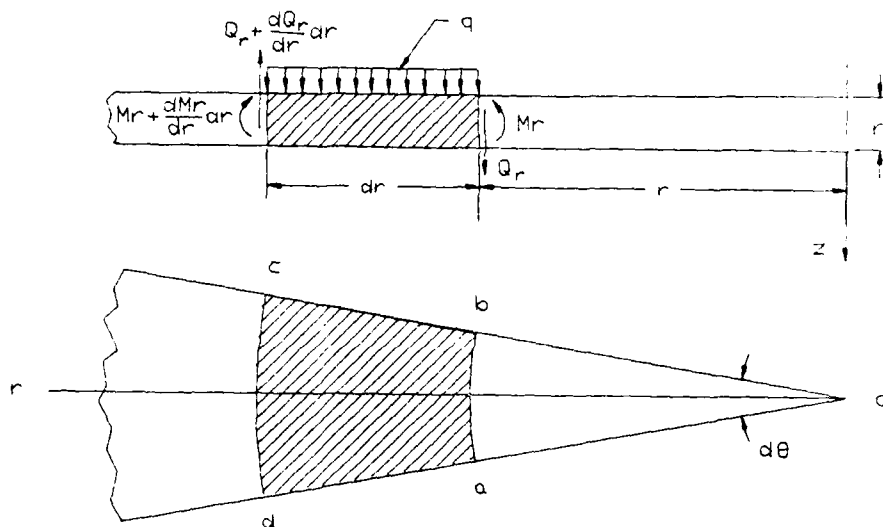


Figure 2-3. Shear forces and moments acting on differential plate element.

Because of symmetry, there are no shear forces acting along edges ad and bc, leaving only the tangential bending moments

$$\left. \begin{array}{l} \text{ad:} \\ \text{bc:} \end{array} \right\} M_{\theta} dr$$

These couples can be resolved into components acting perpendicular to the axis ro (which are equal and opposite and thus cancel), and components acting along the ro axis which have a total magnitude of

$$M_{\theta} dr d\theta$$

Summing up all the couples and neglecting the small change in shear force across the element (i.e. dropping the dQ terms) gives the following equilibrium equation

$$\left(M_r + \frac{dM_r}{dr} dr \right) (r + dr) d\theta - M_r r d\theta - M_{\theta} dr d\theta + Q_r r d\theta dr = 0$$

This equation is further simplified by ignoring higher order terms to give

$$M_r + \frac{dM_r}{dr} r - M_{\theta} + Q_r r = 0 \quad (2-2-1)$$

Also, from equilibrium in the z direction

$$\frac{dQ_r}{dr} r = q \quad (2-2-2)$$

2.3 Differential Equation for Circular Plate Bending

Having found the desired equilibrium equations, the moment curvature relationships of Section 2.1 are now used to develop the governing differential equation for the circular plate bending problem. Specifically, substituting Eq. (2-1-7) into Eq. (2-2-1) results in a third order differential equation to describe the response of a circular plate to lateral loading

$$\frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} = \frac{Q_r}{D} \quad (2-3-1)$$

which can be rewritten to provide an expression for shear

$$Q_r = D \left(\frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} \right) \quad (2-3-2)$$

Differentiating Eq. (2-3-1) with respect to r and using Eq. (2-2) results in a fourth order differential equation in terms of r , w , D , and q

$$\frac{d^4 w}{dr^4} + \frac{2}{r} \frac{d^3 w}{dr^3} - \frac{1}{r^2} \frac{d^2 w}{dr^2} + \frac{1}{r^3} \frac{dw}{dr} = \frac{q}{D} \quad (2-3-3)$$

which can be expressed in a more concise form using the biharmonic operator as

$$\nabla^4 w = \frac{q}{D} \quad (2-3-4)$$

where

$$\nabla^4 = \frac{d^4}{dr^4} + \frac{2}{r} \frac{d^3}{dr^3} - \frac{1}{r^2} \frac{d^2}{dr^2} + \frac{1}{r^3} \frac{d}{dr} \quad (2-3-5)$$

CHAPTER 3

A BOUNDARY INTEGRAL METHOD

This chapter describes a boundary integral method, and how it can be used to provide a general method of solution of circular plate bending problems. A one-dimensional beam bending problem is used to illustrate this method.

3.1 General Formulation

Solution of Eq. (2-3-3) analytically as a boundary value problem by applying known boundary conditions is possible (Timoshenko). However, this approach requires special attention for each plate configuration and load distribution. This difficulty is compounded when considering problems involving plastic behavior which should be solved numerically due to the non-linear behavior in the plastic range.

Therefore, a general method is sought wherein the same procedure can be used to solve a variety of problems, thus lending the problems to computer solution techniques. One such method, adopted for this work to examine circular plate

bending, is the boundary integral method.

Stern and Bezine, working independently, obtained a general formulation for plate bending problems in terms of boundary integrals. These integrals involve displacements, slopes, bending moments, and shears on the plate boundary. Both authors used the following reciprocal identity, which can be derived using Green's identity:

$$D \int_{\Omega} \left(w_G \nabla^4 w - w \nabla^4 w_G \right) d\Omega = \int_{\Gamma} \left(Q_G w + M_G \phi - \phi_G M - w_G Q \right) d\Gamma \quad (3-1-1)$$

where

- Ω - plate domain
- Γ - plate boundary
- D - flexural rigidity
- ∇^4 - biharmonic operator
- w, w_G - deflections
- ϕ, ϕ_G - slopes
- M, M_G - bending moments per unit length
- Q, Q_G - shear forces per unit length

To arrive at the boundary integral equation, w_G is selected as a Green's function.

3.2 Beam Bending Problem

Butterfield has presented a reduced form of Eq. (3-1-1) for the one-dimensional case of beam bending, which is developed in the discussion that follows. As a first step, Butterfield multiplies both sides of the governing differential equation for a beam by a second function and integrates over the beam's length, giving

$$\int_0^L EI \nabla^4 w w_G dx = \int_0^L \lambda w_G dx \quad (3-2-1)$$

where λ is the distributed load per unit length, I is the area moment of inertia of the beam cross section, and the operator ∇^4 is defined for the beam case as

$$\nabla^4 = \frac{d^4}{dx^4}$$

Integrating the left hand side of Eq. (3-2-1) by parts several times, Butterfield develops the equation

$$\int_0^L \left(\lambda w_G - w EI \nabla^4 w_G \right) dx = \left[-w_G Q + \dot{w}_G M - M_G \dot{w} + Q_G w \right]_0^L$$

Recognizing that for the second function

$$EI \nabla^4 w_G = \lambda_G$$

the preceding equation can be rewritten as

$$\int_0^L \left(\lambda w_G - w \lambda_G \right) dx = \left[-w_G Q + \frac{1}{2} M_G^2 - M_G \frac{1}{2} + Q_G w \right]_0^L \quad (3-2-2)$$

To gain some physical insight into Eq. (3-2-2) it is useful to recall Betti's reciprocal work theorem. This theorem states that for an elastic structure subjected to two independent causes (e.g. loads), the total work done by the first cause in moving through the displacements resulting from the second cause is equal to the total work done by the second cause in moving through the displacements produced by the first cause^{<7>}. The theorem, as it applies to our beam bending problem, can be better understood by examining the examples of Figure 3-1.

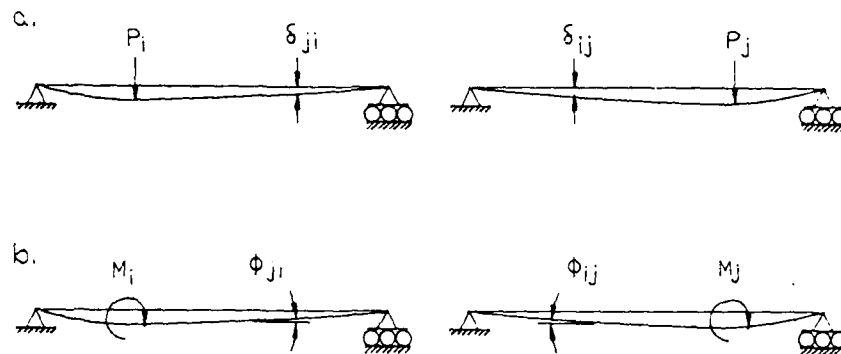


Figure 3-1. Examples of Betti's reciprocal work theorem.

In Figure 3-1a, the product of load P_i and displacement δ_{ij} is equal to the product of load P_j and displacement δ_{ji} . Similarly, in Figure 3-1b, the product of moment M_i and slope ϕ_{ij} is equal to the product of moment M_j and slope ϕ_{ji} .

This concept can be applied to understand Eq. (3-2-2), with the i cause the actual loading or applied moment condition for the system of interest and the j cause the loading or applied moment condition of the second function. In this equation, the λ term represents the external load in the system of interest, which when multiplied by the deflection w_G described by the second function, gives a reciprocal work term. Similarly, the ϕ_G term is a slope for the second function, which when multiplied by the moment M for the system of

interest yields another work term. This concept applies to all the terms of the equation, which are combined according to Betti's theorem of reciprocal work to yield the given equality.

For Eq. (3-2-2) to be useful for solving the beam bending problem, Butterfield chooses as the second function a Green's function which is based on a singular loading condition; that is to say, a point load. If the point load of unit magnitude is applied at either $x=0$ or $x=L$, the second integral term on the left hand side of Eq. (3-2-2) takes on the value of the boundary condition of the beam of interest. Usually, some attention should be given to cases where the integrals become singular.

This yields two equations. However, in general, beam bending problems involve four unknown boundary conditions. For example, if simple supports are specified on both ends, the moments and deflections at the two ends are known to be zero, while the shears and slopes are not known. Therefore, two additional equations are required.

To obtain two more equations, derivatives of the first two equations are taken. This gives an additional Green's function. Boundary conditions can now be obtained by solving the four equations simultaneously. Knowing the boundary conditions, the beam problem can be solved for interior points using Eq. (3-2-2) and appropriate derivatives with the point

load positioned at the location of interest.

The formulation for the beam bending problem outlined above is a direct approach based on Green's identity. Butterfield also develops a boundary integral equation using the more intuitive indirect method. For the circular plate formulation that follows in Chapter 4, only the direct method is used.

CHAPTER 4

DEVELOPMENT OF THE FORMULATION FOR CIRCULAR PLATES

This chapter presents a general method for solving circular plate bending problems in the elastic range using boundary integrals. The resulting integral equations are analagous to those developed by Butterfield for the one-dimensional beam problem, which are discussed in Chapter 3. Treatment of non-linear behavior is discussed in Chapter 5.

4.1 Description of Plate Geometry

Figure 4-1 depicts a symmetrically loaded annular plate, and includes notation that will be referred to throughout the development of the integral equation and the discussion of the selected Green's functions.

Important symbols are defined as follows:

- w, w_G - deflections
 ϕ, ϕ_G - slopes
 M_r, M_{Gr} - radial bending moments per unit length
 $M_\theta, M_{G\theta}$ - tangential bending moments per unit length
 Q_r, Q_{Gr} - radial shear forces per unit length
 $Q_\theta, Q_{G\theta}$ - tangential shear forces per unit length
 a - plate outer radius
 b - plate inner radius
 h - plate thickness
 r - radius of interest
 q - distributed load per unit area

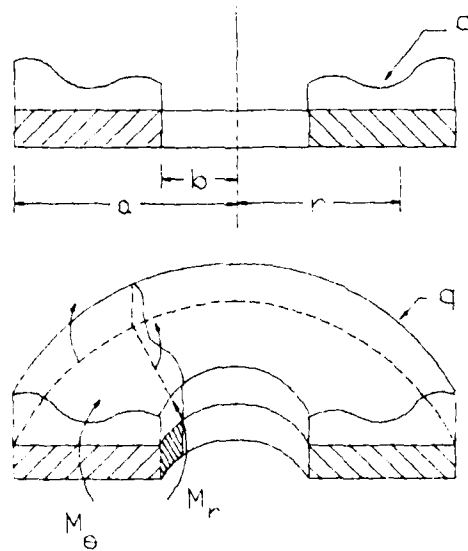


Fig. Symmetrically loaded annular

4.2 Development of Integral Equation

Though the laterally loaded circular plate bending problem is two-dimensional, symmetry reduces the problem to essentially a one-dimensional type problem. Therefore, it is expected that the integral equation developed by Butterfield for the beam case can be adapted to the case of the circular plate. This equation (Eq. (3-2-2)) is repeated below:

$$\int_0^L \left(\lambda w_G - w \lambda_G \right) dx =$$

$$\left[-w_G Q + \bar{t}_G M - M_G \bar{t} + Q_G w \right]_0^L$$

To obtain some indication of what the circular plate integral equation will look like, a non-rigorous method is first used to adapt Eq. (3-2-2) to the circular plate case, followed by a more rigorous mathematical development.

In the non-rigorous approach, a first step is to replace the integral on the left hand side with an area integral and the line load per unit length λ with the load per unit area q . Next, adjustments are made to reciprocal work terms on the boundaries (right hand side of Eq. (3-2-2)). Specifically, the shear terms are replaced with the product of the radial shear per unit length and the total length ($2\pi r$), observing that for symmetrical loading, the tangential shear Q_θ is 0. Moment terms are replaced with radial moments, since there are no tangential moments on the boundaries. These moments must also

be multiplied by the length ($2\pi r$) to give total moment. The resulting expression is

$$\int_0^{2\pi} \int_b^a \left(q w_G - w q_G \right) r dr d\theta =$$

$$\left[2\pi r \left(-w_G Q_r + \frac{1}{2} M_r - M_{Gr} \frac{1}{2} + Q_{Gr} w \right) \right]_b^a \quad (4-2-1)$$

which reduces to

$$\int_b^a \left(q w_G - w q_G \right) r dr =$$

$$\left[r \left(-w_G Q_r + \frac{1}{2} M_r - M_{Gr} \frac{1}{2} + Q_{Gr} w \right) \right]_b^a \quad (4-2-2)$$

So far, it has only been asserted that the integral equation for the circular plate case should have a form similar to Eq. (4-2-2). To obtain the exact integral equation, a direct method is used.

As a starting point for the direct method, consider the following expression:

$$U = \int_b^a \frac{d^2 w_G}{dr^2} \frac{d^2 w}{dr^2} r dr$$

This expression can be written slightly differently as V

$$V = \int_b^a \frac{d^2 w}{dr^2} \frac{d^2 w_G}{dr^2} r \, dr$$

Integrating both expressions by parts yields

$$U = \frac{dw_G}{dr} \frac{d^2 w}{dr^2} r \Big|_b^a - \int_b^a \frac{dw_G}{dr} \left(\frac{1}{r} \frac{d^2 w}{dr^2} + \frac{d^3 w}{dr^3} \right) r \, dr$$

and

$$V = \frac{dw}{dr} \frac{d^2 w_G}{dr^2} r \Big|_b^a - \int_b^a \frac{dw}{dr} \left(\frac{1}{r} \frac{d^2 w_G}{dr^2} + \frac{d^3 w_G}{dr^3} \right) r \, dr$$

Recalling Eq. (2-3-2), it is possible to rewrite these expressions in terms of shear Q_r

$$U = \frac{dw_G}{dr} \frac{d^2 w}{dr^2} r \Big|_b^a - \int_b^a \frac{dw_G}{dr} \frac{Q_r}{D} r \, dr - \int_b^a \frac{1}{r} \frac{dw_G}{dr} \frac{dw}{dr} r \, dr$$

and

$$V = \frac{dw}{dr} \frac{d^2 w_G}{dr^2} r \Big|_b^a - \int_b^a \frac{dw}{dr} \frac{Q_{Gr}}{D} r \, dr - \int_b^a \frac{1}{r} \frac{dw}{dr} \frac{dw_G}{dr} r \, dr$$

Setting U equal to V and canceling some terms gives

$$\left. \frac{dw_G}{dr} \frac{d^2 w}{dr^2} r \right|_b^a - \int_b^a \frac{dw_G}{dr} \frac{Q_r}{D} r dr =$$

$$\left. \frac{dw}{dr} \frac{d^2 w_G}{dr^2} r \right|_b^a - \int_b^a \frac{dw}{dr} \frac{Q_{Gr}}{D} r dr$$

The two integral terms above are integrated by parts, with the operator of Eq. (2-3-5) used to simplify the expressions

$$\int_b^a \frac{dw_G}{dr} \frac{Q_r}{D} r dr = \left. w_G r \frac{Q_r}{D} \right|_b^a - \int_b^a \nabla^4 w w_G r dr$$

$$\int_b^a \frac{dw}{dr} \frac{Q_{Gr}}{D} r dr = \left. w r \frac{Q_{Gr}}{D} \right|_b^a - \int_b^a \nabla^4 w_G w r dr$$

which, when substituted into the previous equation, yield

$$\left. \frac{dw_G}{dr} \frac{d^2 w}{dr^2} r \right|_b^a - \left. w_G r \frac{Q_r}{D} \right|_b^a + \int_b^a \nabla^4 w w_G r dr =$$

$$\left. \frac{dw}{dr} \frac{d^2 w_G}{dr^2} r \right|_b^a - \left. w r \frac{Q_{Gr}}{D} \right|_b^a + \int_b^a \nabla^4 w_G w r dr$$

After some simplification, this expression becomes

$$\int_b^a \left(\nabla^4 w r w_G - \nabla^4 w_G r w \right) dr =$$

$$\left[r \left(w_G \frac{Q_r}{D} - w \frac{Q_{Gr}}{D} + \frac{dw}{dr} \frac{d^2 w_G}{dr^2} - \frac{dw_G}{dr} \frac{d^2 w}{dr^2} \right) \right]_b^a$$

The last equation bears a close resemblance to Eq. (4-2-2), except for sign differences and the fact that terms involving the second derivative of deflection have yet to be replaced by terms involving moment. To accomplish the latter, the following expression is added to the third term of the right hand side of the preceding expression, and subtracted from the fourth term

$$r \frac{v}{r} \frac{dw}{dr} \frac{dw_G}{dr}$$

Through Eq. (2-1-7) this gives

$$\int_b^a \left(\nabla^4 w r w_G - \nabla^4 w_G r w \right) dr = \frac{1}{D} \left[r \left(w_G Q_r - w Q_{Gr} + \dagger M_{Gr} - \dagger_G M_r \right) \right]_b^a \quad (4-2-3)$$

which can be rewritten using Eq. (2-3-4) as

$$\int_b^a \left(q w_G - q_G w \right) r dr = \left[r \left(w_G Q_r - w Q_{Gr} + \dagger M_{Gr} - \dagger_G M_r \right) \right]_b^a \quad (4-2-4)$$

Remarkably, with the exception of a sign difference that is attributable to the difference in sign convention, this expression is identical to Eq. (4-2-2), which was asserted based on the integral equation developed by Butterfield.

4.3 Selection of a Green's Function

As is the case for the beam described in Chapter 3, Eq. (4-2-4) is useful only if w_G is chosen as a Green's function with certain properties. As will become evident later in this chapter, the Green's function must describe a ring loaded plate to solve circular plate bending problems, just as a Green's function describing a point loaded beam was used to solve the beam bending problem.

To solve the beam problem, Butterfield chooses a Green's function describing an infinitely long beam with a point load. It is not essential that a Green's function for an infinite beam be used. In fact, as one would expect from Betti's reciprocal work theorem, a Green's function describing a finite geometry will work as well, provided the function also describes the response to a point load.

Applying Butterfield's method to the circular plate case, a Green's function representing the ring loaded plate shown in Figure 4-2 is adopted. The Green's function and associated derivatives are included in Appendix A. It should be noted that the outer and inner radii of the Green's function (c and d) need not coincide with the outer and inner radii of the actual plate (a and b).

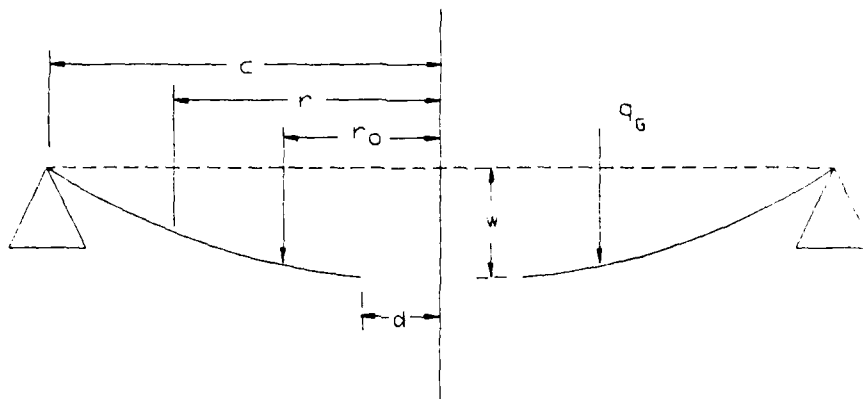


Figure 4-2. Plate geometry for the selected Green's function

That a simple support is chosen for the outer plate radius is not important; a plate clamped at both the inner and outer radii would also have been suitable. What is important is that the loading condition for the Green's function case be a ring load. To simplify calculations, a ring load of magnitude 1 is used.

4.4 Solution of Integral Equation

Having selected a Green's function, the next step is to obtain a form of Eq. (4-2-4) suitable to solve the actual plate bending problem. Rearranging terms, Eq. (4-2-4) can be rewritten as

$$\int_b^a q_G w r dr = \left[r \left(-w_G Q_r + \frac{1}{2} M_r - M_{Gr} \frac{1}{2} + Q_{Gr} w \right) \right]_b^a + \int_b^a q w_G r dr \quad (4-4-1)$$

The ring load may be expressed in terms of the dirac delta function $\delta(r, r_o)$

$$D \nabla^4 w_G = q_G = \delta(r, r_o)$$

Substituting this expression into Eq. (4-4-1) and making use of the following property of a dirac delta function

$$\int_{-\infty}^{\infty} \delta(r, r_o) w r dr = r_o w(r=r_o)$$

gives

$$r_o w(r=r_o) = \left[r \left(-w_G Q_r + \frac{1}{2} M_r - M_{Gr} \frac{1}{2} + Q_{Gr} w \right) \right]_b^a + \int_b^a q w_G r dr \quad (4-4-2)$$

By choosing r_o at radius a , and then at radius b , the deflection w in the left hand side of Eq. (4-4-2) is the deflection at the boundary, which is either a known or unknown boundary condition. Hence, two equations are available to solve for the four unknown boundary conditions.

To obtain two more equations, Eq. (4-4-2) is differentiated with respect to r_o . This yields the second Green's function, which is evaluated with r_o at a and then again at b . The second Green's function corresponds to the slope of the plate, which at $r=a$ and $r=b$ is again either a known or an unknown boundary condition. There is thus now a complete set of four equations to solve for the four unknown boundary conditions.

Eq. (4-4-2) can be rewritten as

$$w(r=r_o) = \left[\frac{r}{r_o} \left(-w_G Q_r + \frac{1}{2} M_r - M_{Gr} \frac{1}{2} + Q_{Gr} w \right) \right]_b^a + \frac{1}{r_o} \int_b^a q w_G r dr \quad (4-4-3)$$

Differentiating this expression with respect to r_o gives

$$\frac{dw(r=r_o)}{dr_o} = \left[\frac{r}{r_o} \left(-\frac{dw_G}{dr_o} Q_r + \frac{d}{dr_o} \frac{1}{2} M_r - \frac{dM_{Gr}}{dr_o} \frac{1}{2} + \frac{dQ_{Gr}}{dr_o} w \right) \right]_b^a + \frac{1}{r_o} \int_b^a q \frac{dw_G}{dr} r dr - \frac{w(r=r_o)}{r_o} \quad (4-4-4)$$

Eqs. (4-4-3) and (4-4-4), evaluated both for $r_o=a$ and $r_o=b$, give rise to a system of four equations that when solved simultaneously, yield the four unknown boundary conditions. In matrix form, these equations can be written as

$$\begin{bmatrix} aw(a) \\ bw(b) \\ -a\dot{\phi}(a) \\ -b\dot{\phi}(b) \end{bmatrix} = a \begin{bmatrix} -w_G(a,a) & \dot{\phi}_G(a,a) & -M_{Gr}(a,a) & Q_{Gr}(a,a) \\ -w_G(a,b) & \dot{\phi}_G(a,b) & -M_{Gr}(a,b) & Q_{Gr}(a,b) \\ -w_G^1(a,a) & \dot{\phi}_G^1(a,a) & -M_{Gr}^1(a,a) & Q_{Gr}^1(a,a) \\ -w_G^1(a,b) & \dot{\phi}_G^1(a,b) & -M_{Gr}^1(a,b) & Q_{Gr}^1(a,b) \end{bmatrix} \begin{bmatrix} Q_r(a) \\ M_r(a) \\ \dot{\phi}(a) \\ w(a) \end{bmatrix} \\
- b \begin{bmatrix} -w_G(b,a) & \dot{\phi}_G(b,a) & -M_{Gr}(b,a) & Q_{Gr}(b,a) \\ -w_G(b,b) & \dot{\phi}_G(b,b) & -M_{Gr}(b,b) & Q_{Gr}(b,b) \\ -w_G^1(b,a) & \dot{\phi}_G^1(b,a) & -M_{Gr}^1(b,a) & Q_{Gr}^1(b,a) \\ -w_G^1(b,b) & \dot{\phi}_G^1(b,b) & -M_{Gr}^1(b,b) & Q_{Gr}^1(b,b) \end{bmatrix} \begin{bmatrix} Q_r(b) \\ M_r(b) \\ \dot{\phi}(b) \\ w(b) \end{bmatrix} \\
+ \begin{bmatrix} \int_b^a q w_G(r,a) r dr \\ \int_b^a q w_G(r,b) r dr \\ \int_b^a q w_G^1(r,a) r dr - w(a) \\ \int_b^a q w_G^1(r,b) r dr - w(b) \end{bmatrix} \quad (4-4-5)$$

where

$$\left(\begin{array}{c} \\ \end{array} \right) (s,t) = \left(\begin{array}{c} \\ \end{array} \right) (r=s; r_0=t)$$

and

$$\left(\begin{array}{c} \\ \end{array} \right)' = \frac{d}{dr_0} \left(\begin{array}{c} \\ \end{array} \right)$$

Using Gauss elimination, the preceding system of equations is solved for the four unknown boundary conditions.

The final step is to calculate deflection, slope, moment and shear across the plate. Deflection and slope are found

using Eqs. (4-4-3) and (4-4-4) by substituting in the calculated boundary conditions and evaluating the expressions at the value of r_o of interest. Moment and shear are determined using Eqs. (2-1-7) and (2-3-2), which require calculation of the second and third derivatives of w with respect to r_o . Differentiating Eq. (4-4-4) results in the desired expressions

$$\begin{aligned} \frac{d^2 w(r=r_o)}{dr_o^2} = & \left[\frac{r}{r_o} \left(- \frac{d^2 w_G}{dr_o^2} Q_r + \frac{d^2 \frac{1}{2} G}{dr_o^2} M_r - \frac{d^2 M_{Gr}}{dr_o^2} \frac{1}{2} + \frac{d^2 Q_{Gr}}{dr_o^2} w \right) \right]_b^a \\ & + \frac{1}{r_o} \int_b^a q \frac{d^2 w_G}{dr_o^2} r dr - \frac{2}{r_o} \frac{dw(r=r_o)}{dr_o} \quad (4-4-6) \end{aligned}$$

and

$$\begin{aligned} \frac{d^3 w(r=r_o)}{dr_o^3} = & \left[\frac{r}{r_o} \left(- \frac{d^3 w_G}{dr_o^3} Q_r + \frac{d^3 \frac{1}{2} G}{dr_o^3} M_r - \frac{d^3 M_{Gr}}{dr_o^3} \frac{1}{2} + \frac{d^3 Q_{Gr}}{dr_o^3} w \right) \right]_b^a \\ & + \frac{1}{r_o} \int_b^a q \frac{d^3 w_G}{dr_o^3} r dr - \frac{3}{r_o} \frac{d^2 w(r=r_o)}{dr_o^2} \quad (4-4-7) \end{aligned}$$

making a complete solution for the circular plate bending problem in the elastic range possible.

CHAPTER 5

NON-LINEAR BEHAVIOR

This chapter develops a boundary integral formulation for circular plate bending problems that includes non-linear material behavior. Specifically, the boundary integral equation developed in Chapter 4 for the elastic plate is modified to include plastic moment terms to account for the plastic behavior. An incremental load method similar to that used by Moshaiov and Vorus can then be applied to arrive at a solution to the plate bending problem once yielding has begun.

5.1 Plastic Stress-Strain Relationships

When considering plasticity, strain can be thought of as having two components: an elastic component and a plastic component. In cylindrical coordinates, which is applicable to the two-dimensional circular plate case, the total strain ϵ can be expressed in terms of these two components as

$$\begin{aligned}\epsilon_r &= \epsilon_r^e + \epsilon_r^p \\ \epsilon_\theta &= \epsilon_\theta^e + \epsilon_\theta^p\end{aligned}\tag{5-1-1}$$

where the superscripts e and p refer to the elastic and plastic strain components, respectively.

Eq. (2-1-4), which provides the relationship between stress and strain for a linearly elastic material, can thus be rewritten in terms of the total strain and plastic strain as follows:

$$\begin{aligned}\sigma_r &= \frac{E}{(1-\nu^2)} \left[\left(\epsilon_r - \epsilon_r^p \right) + \nu \left(\epsilon_\theta - \epsilon_\theta^p \right) \right] \\ \sigma_\theta &= \frac{E}{(1-\nu^2)} \left[\left(\epsilon_\theta - \epsilon_\theta^p \right) + \nu \left(\epsilon_r - \epsilon_r^p \right) \right]\end{aligned}\tag{5-1-2}$$

5.2 Plastic Moment-Curvature Relationships

To develop the moment-curvature relationships with plasticity, Eq. (5-1-2) is substituted into Eq. (2-1-3) to give

$$M_r = \int_{-h/2}^{h/2} \frac{E}{(1-\nu^2)} \left[\left(\epsilon_r - \epsilon_r^p \right) + \nu \left(\epsilon_\theta - \epsilon_\theta^p \right) \right] z \, dz$$

$$M_{\theta} = \int_{-h/2}^{h/2} \frac{E}{(1-\nu^2)} \left[\left(\epsilon_{\theta} - \epsilon_{\theta}^p \right) + \nu \left(\epsilon_r - \epsilon_r^p \right) \right] z \, dz$$

These expressions can be rewritten as

$$M_r = \frac{E}{(1-\nu^2)} \int_{-h/2}^{h/2} \left(\epsilon_r + \nu \epsilon_{\theta} \right) z \, dz - M_r^p \quad (5-2-1)$$

$$M_{\theta} = \frac{E}{(1-\nu^2)} \int_{-h/2}^{h/2} \left(\epsilon_{\theta} + \nu \epsilon_r \right) z \, dz - M_{\theta}^p$$

where the radial and tangential plastic moments, M_r^p and M_{θ}^p , are defined as

$$M_r^p = \frac{E}{(1-\nu^2)} \int_{-h/2}^{h/2} \left(\epsilon_r^p + \nu \epsilon_{\theta}^p \right) z \, dz \quad (5-2-2)$$

$$M_{\theta}^p = \frac{E}{(1-\nu^2)} \int_{-h/2}^{h/2} \left(\epsilon_{\theta}^p + \nu \epsilon_r^p \right) z \, dz$$

Recalling the relationships for total strain of Eq. (2-1-6) and performing the integration over the plate thickness yield the moment-curvature relationship for a circular plate with

plasticity

$$M_r = -D \left(\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right) - M_r^P \quad (5-2-3)$$

$$M_\theta = -D \left(\frac{1}{r} \frac{dw}{dr} + \frac{\nu}{r} \frac{d^2 w}{dr^2} \right) - M_\theta^P$$

5.3 Governing Differential Equation With Plasticity

Chapter 2 develops the governing differential equation for a circular plate in the elastic range by substituting the elastic moment-curvature relationships into the elastic equilibrium equation. In this section, a similar approach is used to develop the governing differential equation with plasticity. The difference here is that the moment-curvature relationships (Eq. (5-2-3)) include plastic terms.

Substituting Eq. (5-2-3) into the equilibrium equation for a circular plate (Eq. (2-3-1)) gives

$$\begin{aligned} \frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} + \frac{1}{r D} \left(M_r^P + \frac{dM_r^P}{dr} r - M_\theta^P \right) \\ = \frac{\tilde{Q}}{D} r \end{aligned} \quad (5-3-1)$$

where the tilde is used to distinguish the fact that the shear includes plastic effects. Differentiating this expression once with respect to r gives the fourth order governing differential

equation for the plate with plasticity

$$\frac{d^4 w}{dr^4} + \frac{2}{r} \frac{d^3 w}{dr^3} - \frac{1}{r^2} \frac{d^2 w}{dr^2} + \frac{1}{r^3} \frac{dw}{dr} = \frac{q}{D} - \frac{1}{D} \left(\frac{2}{r} \frac{dM_r^P}{dr} + \frac{d^2 M_r^P}{dr^2} - \frac{1}{r} \frac{dM_\theta^P}{dr} \right)$$

which can be rewritten using the operator defined in Eq. (2-3-5) as

$$\nabla^4 w = \frac{q}{D} - \frac{1}{D} \left(\frac{2}{r} \frac{dM_r^P}{dr} + \frac{d^2 M_r^P}{dr^2} - \frac{1}{r} \frac{dM_\theta^P}{dr} \right) \quad (5-3-2)$$

This equation is identical to the equilibrium equation for the elastic case with the exception of the terms that include derivatives of the plastic moments. Note that these terms may be thought of as additional pseudo loads that must be combined with the actual load to account for the plastic behavior.

5.4 Boundary Integral Formulation With Plasticity

In the preceding chapter, the governing differential equation is used to develop a boundary integral formulation for a circular plate in the elastic case. This section presents a similar approach to develop a boundary integral formulation for the plastic case.

As a starting point, Eqs. (2-3-4) and (5-3-2) are substituted into Eq. (4-2-3) to give

$$\int_b^a q_G w r dr = \left[r \left(-w_G Q_r + \dot{\phi}_G M_r - M_{Gr} \dot{\phi} + Q_{Gr} w \right) \right]_b^a$$

(5-4-1)

$$+ \int_b^a q w_G r dr - \int_b^a \left(\frac{2}{r} \frac{dM_r^P}{dr} + \frac{d^2 M_r^P}{dr^2} - \frac{1}{r} \frac{dM_\theta^P}{dr} \right) w_G r dr$$

This equation is not yet in a form that is usable for solving the circular plate bending problem. In the first place, there are several terms that include derivatives of the radial or tangential plastic moments, which are not generally known across the plate. Secondly, the shear and moment terms Q_r and M_r evaluated at the boundaries are not the actual shears and moments, since they were developed based on elastic considerations only.

To resolve these concerns, the integral term involving plastic moments in Eq. (5-4-1) is integrated by parts as follows:

$$\begin{aligned}
& \int_b^a w_G \left(2 \frac{dM_r^P}{dr} + r \frac{d^2 M_r^P}{dr^2} - \frac{dM_\theta^P}{dr} \right) dr = \\
& \left[M_G \left(M_r^P + r \frac{dM_r^P}{dr} - M_\theta^P \right) \right]_b^a \\
& - \int_b^a \frac{dw_G}{dr} \left(M_r^P + r \frac{dM_r^P}{dr} - M_\theta^P \right) dr
\end{aligned}$$

Integrating by parts a second time and collecting terms gives

$$\begin{aligned}
& \int_b^a w_G \left(2 \frac{dM_r^P}{dr} + r \frac{d^2 M_r^P}{dr^2} - \frac{dM_\theta^P}{dr} \right) dr = \\
& \left[w_G \left(M_r^P + r \frac{dM_r^P}{dr} - M_\theta^P \right) - r \frac{dw_G}{dr} M_r^P \right]_b^a \\
& + \int_b^a \left(\frac{dw_G}{dr} M_\theta^P + r \frac{d^2 w_G}{dr^2} M_r^P \right) dr
\end{aligned}$$

Using the above relationship and choosing a Green's function that describes a ring loaded plate as is done for Eq. (4-4-2), Eq. (5-4-1) becomes

$$\begin{aligned}
r_o w(r=r_o) &= \left[r \left(-w_G Q_r + \frac{1}{r} M_r - M_{Gr} \frac{1}{r} + Q_{Gr} w \right) \right]_b^a \\
&- \left[w_G \left(M_r^P + r \frac{dM_r^P}{dr} - M_\theta^P \right) - r \frac{dw_G}{dr} M_r^P \right]_b^a \\
&+ \int_b^a q w_G r dr - \int_b^a \left(\frac{dw_G}{dr} M_\theta^P + \frac{d^2 w_G}{dr^2} r M_r^P \right) dr
\end{aligned}$$

Recall that Q_r and M_r represent shear and moment for the elastic case. However, they are not the actual shear and moment for the plastic case. With plasticity, the actual shears and moments are combinations of elastic and plastic terms. From Eq. (5-3-1), the shear relationship with plasticity is

$$\tilde{Q}_r = Q_r + \frac{1}{r} \left(M_r^P + r \frac{dM_r^P}{dr} - M_\theta^P \right)$$

and from Eq. (5-2-3), the moment relationship with plasticity is

$$\tilde{M}_r = M_r + M_r^P$$

where, as with the case for shear, the tilde is used to distinguish the fact that the moment term includes plastic effects. When substituted into the preceding expression, these expressions give the desired boundary integral formulation for the circular plate bending problem

$$\begin{aligned}
w(r=r_0) = & \left[\frac{r}{r_0} \left(-w_G \tilde{Q}_r + \tilde{M}_G \tilde{M}_r - M_{Gr} \tilde{M} + Q_{Gr} w \right) \right]_b^a \\
& + \frac{1}{r_0} \int_b^a q w_G r dr \\
& - \frac{1}{r_0} \int_b^a \left(\frac{dw_G}{dr} M_\theta^p + \frac{d^2 w_G}{dr^2} r M_r^p \right) dr \quad (5-4-2)
\end{aligned}$$

By evaluating Eq. (5-4-2) with the ring load positioned both at $r_0=a$ and $r_0=b$, two equations are obtained to solve for the unknown boundary conditions. Eq. (5-4-2) is differentiated with respect to r_0 to obtain a second Green's function. This, via Eq. (2-1-1), gives an expression for slope of the actual plate. The second Green's function is evaluated with the ring load positioned at $r_0=a$ and $r_0=b$, to provide the remaining two equations.

Differentiating Eq. (5-4-2) with respect to r_0 gives

$$\begin{aligned}
\frac{dw(r=r_o)}{dr_o} = & \left[\frac{r}{r_o} \left(-\frac{dw_G}{dr_o} \tilde{Q}_r + \frac{d\tilde{Q}_G}{dr_o} \tilde{M}_r - \frac{dM_{Gr}}{dr_o} \tilde{r} + \frac{dQ_{Gr}}{dr_o} w \right) \right]_b^a \\
& + \frac{1}{r_o} \int_b^a q \frac{dw_G}{dr_o} r dr - \frac{w(r=r_o)}{r_o} \\
& - \frac{1}{r_o} \int_b^a \left[\frac{d}{dr_o} \left(\frac{dw_G}{dr} \right) M_\theta^p + \frac{d}{dr_o} \left(\frac{d^2 w_G}{dr^2} \right) r M_r^p \right] dr \quad (5-4-3)
\end{aligned}$$

In matrix form, the system of equations to be solved for the four unknown boundary conditions for the non-linear case can be written using the notation of Eq. (4-4-5) as follows:

$$\begin{aligned}
& \begin{bmatrix} aw(a) \\ bw(b) \\ -a\phi(a) \\ -b\phi(b) \end{bmatrix} = a \begin{bmatrix} -w_G(a,a) & \phi_G(a,a) & -M_{Gr}(a,a) & Q_{Gr}(a,a) \\ -w_G(a,b) & \phi_G(a,b) & -M_{Gr}(a,b) & Q_{Gr}(a,b) \\ -w_G^1(a,a) & \phi_G^1(a,a) & -M_{Gr}^1(a,a) & Q_{Gr}^1(a,a) \\ -w_G^1(a,b) & \phi_G^1(a,b) & -M_{Gr}^1(a,b) & Q_{Gr}^1(a,b) \end{bmatrix} \begin{bmatrix} \tilde{Q}_r(a) \\ \tilde{M}_r(a) \\ \phi(a) \\ w(a) \end{bmatrix} \\
& - b \begin{bmatrix} -w_G(b,a) & \phi_G(b,a) & -M_{Gr}(b,a) & Q_{Gr}(b,a) \\ -w_G(b,b) & \phi_G(b,b) & -M_{Gr}(b,b) & Q_{Gr}(b,b) \\ -w_G^1(b,a) & \phi_G^1(b,a) & -M_{Gr}^1(b,a) & Q_{Gr}^1(b,a) \\ -w_G^1(b,b) & \phi_G^1(b,b) & -M_{Gr}^1(b,b) & Q_{Gr}^1(b,b) \end{bmatrix} \begin{bmatrix} \tilde{Q}_r(b) \\ \tilde{M}_r(b) \\ \phi(b) \\ w(b) \end{bmatrix} \\
& + \begin{bmatrix} \int_b^a q w_G(r,a) r dr \\ \int_b^a q w_G(r,b) r dr \\ \int_b^a q w_G^1(r,a) r dr - w(a) \\ \int_b^a q w_G^1(r,b) r dr - w(b) \end{bmatrix} \quad (5-4-4) \\
& - \begin{bmatrix} \int_b^a \left(\frac{dw_G(r,a)}{dr} M_\theta^p + \frac{d^2 w_G(r,a)}{dr^2} r M_r^p \right) dr \\ \int_b^a \left(\frac{dw_G(r,b)}{dr} M_\theta^p + \frac{d^2 w_G(r,b)}{dr^2} r M_r^p \right) dr \\ \int_b^a \left[\frac{d}{dr_0} \left(\frac{dw_G(r,a)}{dr} \right) M_\theta^p + \frac{d}{dr_0^2} \left(\frac{d^2 w_G(r,a)}{dr^2} \right) r M_r^p \right] dr \\ \int_b^a \left[\frac{d}{dr_0} \left(\frac{dw_G(r,b)}{dr} \right) M_\theta^p + \frac{d}{dr_0^2} \left(\frac{d^2 w_G(r,b)}{dr^2} \right) r M_r^p \right] dr \end{bmatrix}
\end{aligned}$$

Observe that Eq. (5-4-4) is identical to Eq. (4-4-5) except for the additional moment terms in the last matrix of Eq. (5-4-4). These additional terms account for the plastic behavior.

To find moments and shears across the plate, expressions for the second and third derivatives of deflection with respect to r_o are required. These expressions follow:

$$\begin{aligned} \frac{d^2 w(r=r_o)}{dr_o^2} &= \left[\frac{r}{r_o} \left(-\frac{d^2 w_G}{dr_o^2} \tilde{Q}_r + \frac{d^2 \tilde{Q}_G}{dr_o^2} \tilde{M}_r - \frac{d^2 M_{Gr}}{dr_o^2} \tilde{Q} + \frac{d^2 Q_{Gr}}{dr_o^2} \tilde{w} \right) \right]_b^a \\ &+ \frac{1}{r_o} \int_b^a q \frac{d^2 w_G}{dr_o^2} r dr - \frac{2}{r_o} \frac{dw(r=r_o)}{dr_o} \\ &- \frac{1}{r_o} \int_b^a \left[\frac{d^2 \left(\frac{dw_G}{dr} \right)}{dr_o^2} M_\theta^p + \frac{d^2 \left(\frac{d^2 w_G}{dr^2} \right)}{dr_o^2} r M_r^p \right] dr \quad (5-4-5) \end{aligned}$$

$$\begin{aligned}
\frac{d^3 w(r=r_o)}{dr_o^3} = & \left[\frac{r}{r_o} \left(- \frac{d^3 w_G}{dr_o^3} \tilde{Q}_r + \frac{d^3 \phi_G}{dr_o^3} \tilde{M}_r - \frac{d^3 M_{Gr}}{dr_o^3} \phi + \frac{d^3 Q_{Gr} w}{dr_o^3} \right) \right]_b^a \\
& + \frac{1}{r_o} \int_b^a q \frac{d^3 w_G}{dr_o^3} r dr - \frac{3}{r_o} \frac{d^2 w(r=r_o)}{dr_o^2} \\
& - \frac{1}{r_o} \int_b^a \left[\frac{d^3 \left(\frac{dw_G}{dr} \right)}{dr_o^3} M_\theta^p + \frac{d^3 \left(\frac{d^2 w_G}{dr^2} \right)}{dr_o^3} r M_r^p \right] dr \quad (5-4-6)
\end{aligned}$$

CHAPTER 6

NUMERICAL SOLUTION

This chapter describes the numerical procedure that was used to solve the circular plate bending problem in both the elastic and plastic ranges. The incremental load approach used in the plastic range is first described. A brief description of the computer program that was developed based on the equations of Chapters 4 and 5 follows, including a discussion of some aspects of numerical methods used in this program.

6.1 Incremental Load Method

Once plastic deformation starts to occur in the plate, the linear relation between stress and strain is not valid. As discussed by Mendelson⁽⁸⁾, strains in the plastic range are no longer uniquely determined by the stresses, and in fact depend on the loading history. Therefore, it is not possible to use the relationships developed in Chapter 5 to directly arrive at a plate bending solution given a loading condition outside the elastic range. Rather, an incremental approach must be used wherein the load is increased in small steps once yielding

occurs at any point in the plate, and the complete stress state for the plate is determined before load again is increased.

The incremental load method used for this analysis is based on that adopted by Moshaiov and Vorus, and is frequently seen in finite element solutions of structural analysis problems. The principal steps of this method are outlined below:

1. Using the plate parameters and loading conditions for the bending problem at hand, an elastic solution is obtained with the equations developed in Chapter 4.

2. The load at which the onset of yielding will occur in the plate is calculated based on the stress state determined in Step 1. The selected yield criterion is the von Mises criterion, which for the symmetrically loaded circular plate case has the form:

$$\sigma_e = \left(\sigma_r^2 - \sigma_r \sigma_\theta + \sigma_\theta^2 \right)^{\frac{1}{2}}$$

where σ_e is the equivalent, or effective, stress which represents the von Mises yield surface. Yielding will occur when σ_e equals or exceeds the uniaxial yield stress (S_y).

3. Unknown boundary conditions at yielding are determined using Eq. (5-4-4), with plastic moments initially

set equal to zero. Total strains at predetermined integration points are calculated using Eqs. (2-1-6), (5-4-3) and (5-4-5). From Eq. (2-1-4), the stress state at the integration points at yielding is determined, and is stored to be later updated as load is increased. Also stored for later updating are the deflections, slopes, moments and shears across the plate.

4. The load which caused yield is then increased by a small incremental amount. The increments of the total strains resulting from this incremental load are calculated using Eq. (2-1-6) in an incremental form, from which elastic strains can be calculated by means of Eq. (5-1-1).

5. Using Eq. (2-1-4), the elastic stress increment corresponding to the applied incremental load is calculated. This incremental stress is added to the stress stored in Step 3 to give the total stress at the end of the load increment. A check is made throughout the plate using the yield criterion of Step 2 to determine where yielding has occurred.

6. In those plate regions where yielding has occurred, the plastic strain increment (that is, the plastic strain due to the incremental load) is calculated. Only materials that exhibit elastic-perfectly plastic behavior as depicted in Figure 6-1 have been examined. However, the method can be used with other material behavior, such as strain hardening. For materials exhibiting elastic-perfectly plastic behavior, the

plastic strain increment $\delta\epsilon_p$ is the total incremental strain determined from Eq. (2-1-6). This is shown in Figure 6-1.

7. The plastic moments are determined using Eq. (5-2-2).

8. Steps 3 through 6 are repeated, with the plastic moments calculated in Step 7 used as an input to Eq. (5-4-4). Step 7 is repeated and the plastic moments compared to the moments calculated previously in Step 7. Iterations are performed as necessary until these values converge, according to a predetermined convergence criteria.

9. After convergence of plastic moments is achieved in Step 8, stresses, plastic strains, deflections, slopes, moments and shears throughout the plate are updated.

10. Another increment of load is applied, and Steps 3 through 9 above are repeated, using the plastic moment from the previous load step as an initial estimate for plastic moment.

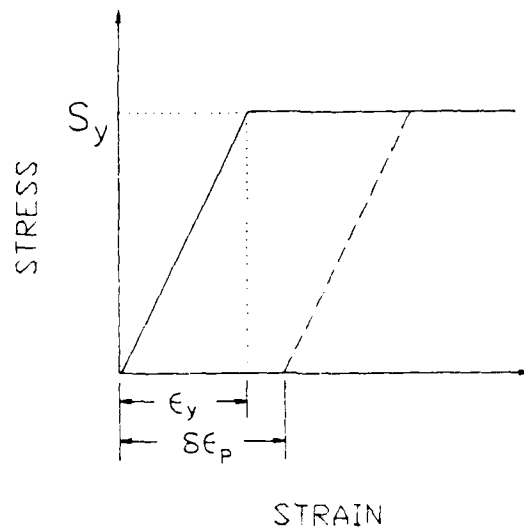


Figure 6.1 Uniaxial stress-strain curve for a perfectly plastic material.

6.2 Computer Program Description

Computer programs written in the Fortran 77 language have been developed to solve the circular plate bending problem and are included as Appendix B. Three programs, supported by nine subroutines, are used.

The programs INPUT and LOAD create data files which are used by the program MAIN to solve the bending problem. INPUT and LOAD prompt the user for information describing the problem to be solved, such as material properties, plate geometry, boundary conditions and plate loading conditions. In addition, certain adjustable parameters are input, which include step

sizes for the load increments, number of integration increments, and the desired percentage for convergence of the plastic moment increments. The program MAIN determines the plate bending solution, both in the elastic and plastic ranges.

A flow diagram outlining the major elements and overall logic path of this program is shown in Figure 6-2.

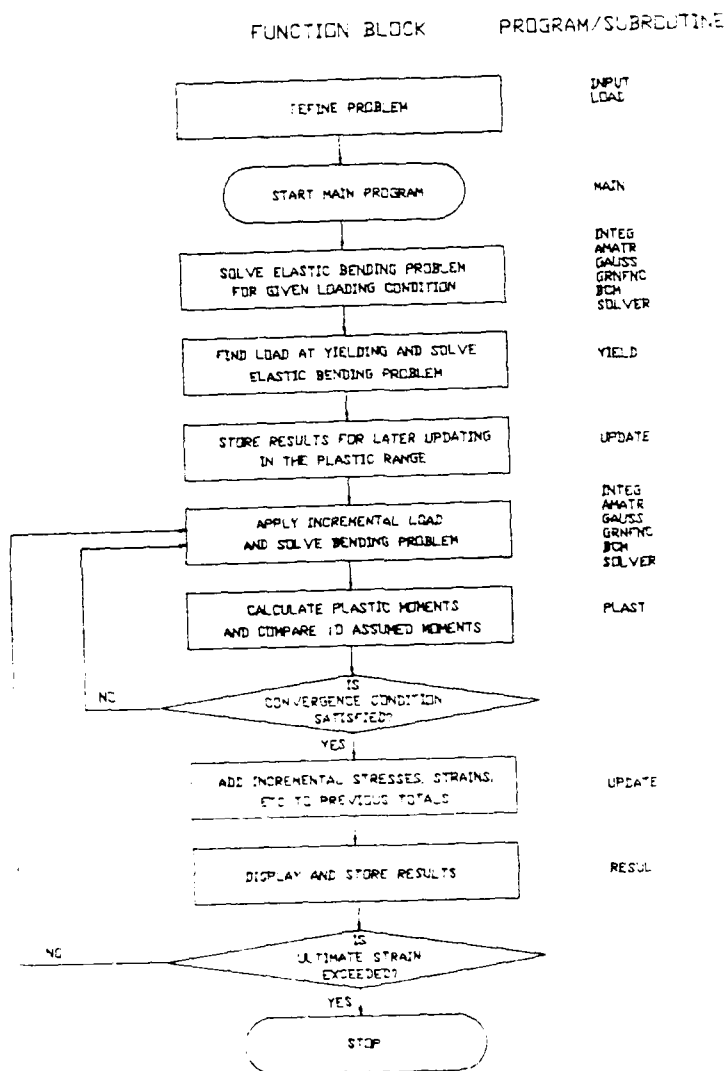


Figure 6-2. Computer program flow diagram.

6.3 Numerical Methods

Eqs. (5-2-1), (5-4-4), (5-4-5) and (5-4-6) require integrations to be performed across the plate. These integrations are accomplished by a trapezoidal rule numerical integration scheme. It was found that 10 increments across both the plate radius and the plate half thickness gave satisfactory results.

To ensure convergence of plastic moments, a root mean square average of the the plastic moments at each station across the plate is calculated and compared to the root mean square average from the previous iteration. An agreement of less than 0.1% was used for the analysis work for which results appear in Chapter 7.

CHAPTER 7

RESULTS AND DISCUSSION

This chapter presents results of analyses performed on three different plate configurations using the computer program described in the preceding chapter. Results in the elastic range for all three cases are compared with the analytic solution from Roark^{<9>}, and for one case in the plastic range where published results using another solution method are available. Descriptions of the case studies are given in Sections 7.1 through 7.3. A discussion of the results is given in Section 7.4.

7.1 Simply Supported Elasto-Plastic Annular Plate

The annular plate shown in Figure 7-1 is first examined. The plate is simply supported at both the inner and outer radii, and is subjected to a uniform lateral load. A material exhibiting elastic-perfectly plastic behavior as depicted in Figure 6-1 is used. The yield stress (S_y) is 16 ksi and the Young's modulus is 10×10^3 ksi. Material properties and plate dimensions were selected to allow comparison with results

obtained using a finite element method by Armen et al⁽¹⁰⁾.

Plate deflection at yielding is shown in Figure 7-2, which also includes results from the analytic solution given by Roark for comparison. Plate deflections for the present solution at several loads in the plastic range are shown in Figure 7-3, as well as results obtained by Armen et al using a finite element method of analysis.

Figure 7-4 shows the plastic zones in the plate at the three load conditions depicted in Figure 7-3. Results obtained by Armen et al are also included.

$$S_y = 16 \text{ KSI}$$

$$E = 10 \times 10^3 \text{ KSI}$$

$$\nu = 0.24$$

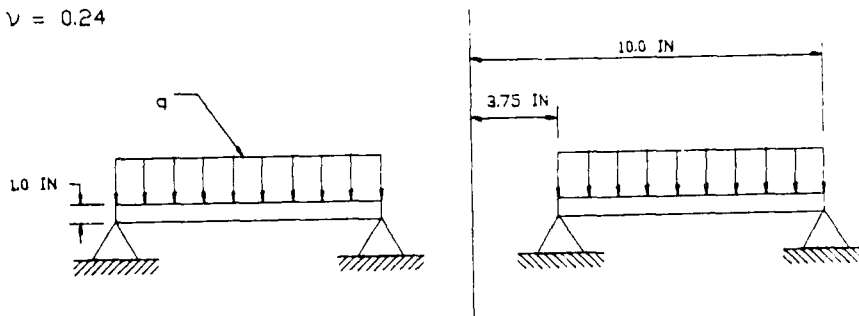


Figure 7-1. Simply supported annular plate problem description.

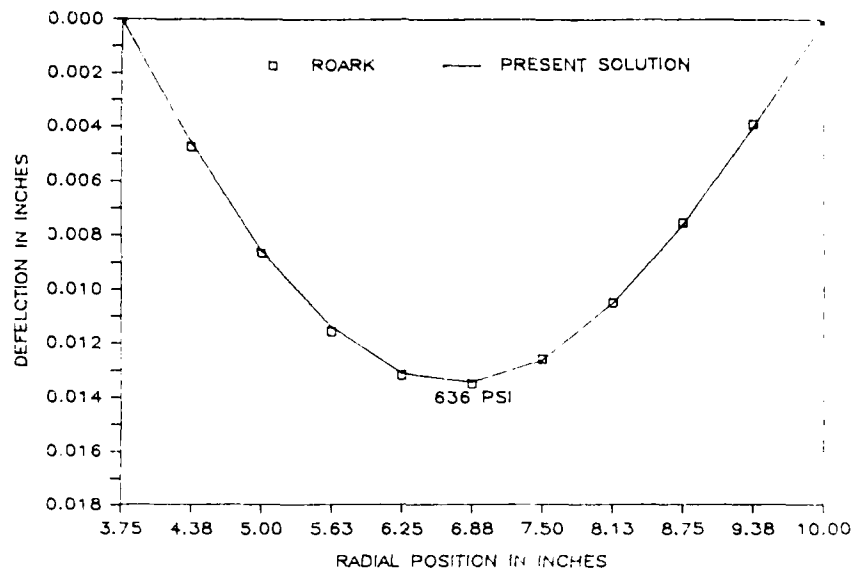


Figure 7-2. Deflection of simply supported annular plate in the elastic range.

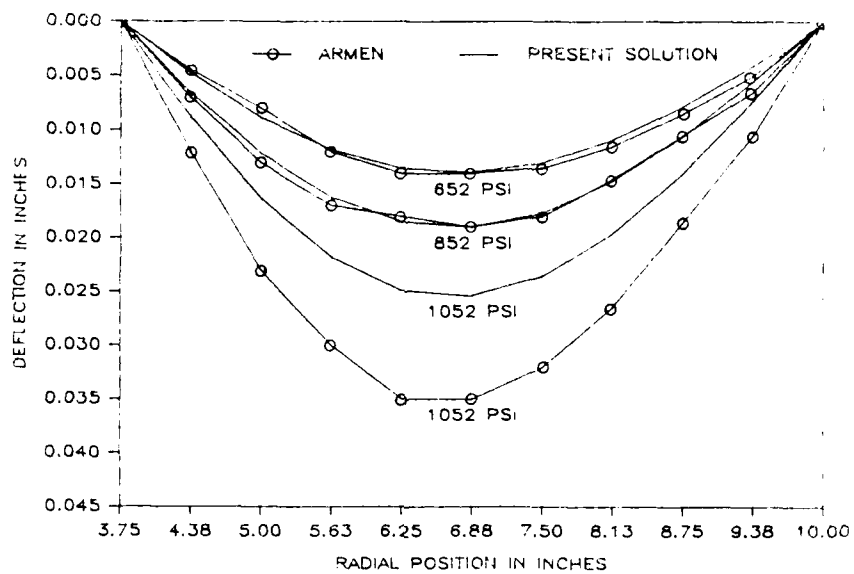


Figure 7-3. Deflection of simply supported annular plate in the plastic range.

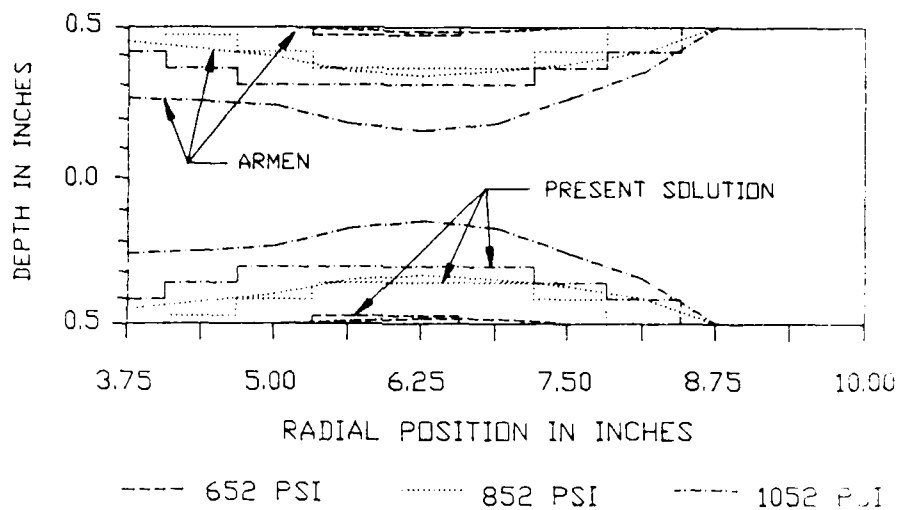


Figure 7-4. Plastic zones in simply supported annular plate.

7.2 Clamped Elasto-Plastic Annular Plate

The annular plate shown in Figure 7-5 is next examined. The plate is clamped at both the inner and outer radii, and is again subjected to a uniform lateral load. A material exhibiting the same properties as described in Section 7.1 is used. For the clamped annular plate case, results obtained using other methods could not be found for the plastic range. Consequently, material properties and plate dimensions were selected to allow a qualitative comparison with results for the simply supported plate of Section 7.1

Plate deflections at the onset of yielding are shown in Figure 7-6, and at several loads in the plastic range in Figure

7-7. The deflection at yielding using the analytic solution given by Roark is shown in Figure 7-6 for comparison. Finally, Figure 7-8 shows the plastic zones in the plate at the three load conditions depicted in Figure 7-7.

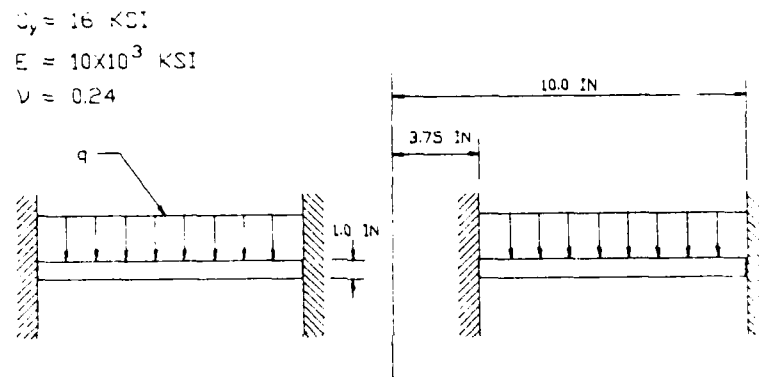


Figure 7-5. Clamped annular plate problem description

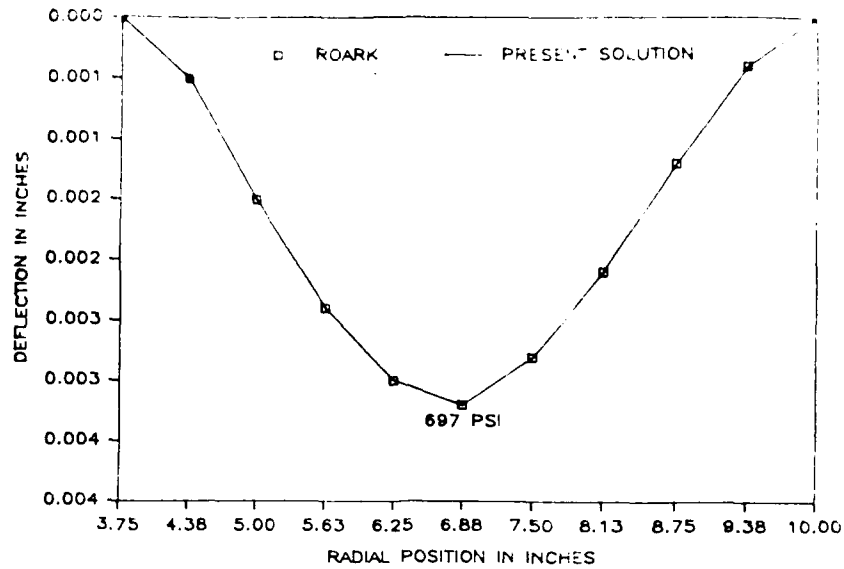


Figure 7-6. Deflection of clamped annular plate at the onset of yielding

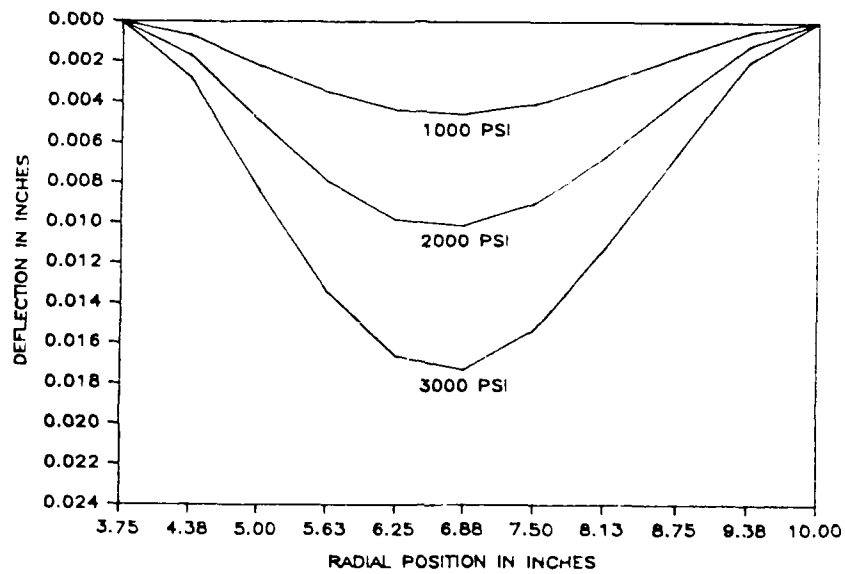


Figure 7-7. Deflection of clamped annular plate in the plastic range

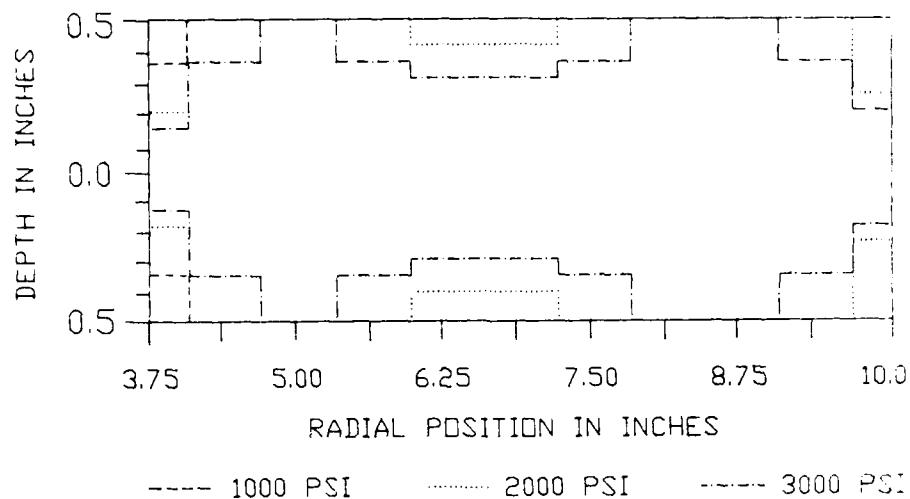


Figure 7-8. Plastic zones in clamped annular plate.

7.3 Simply Supported/Guided Elasto-Plastic Annular Plate

The third annular plate to be examined is shown in Figure 7-8. The plate is simply supported at the outer radius, with the inner radius guided. The plate is again subject to a uniform lateral load. A material exhibiting the same properties described in Section 7-1 is assumed.

For this case, results obtained using other methods could again not be found for the plastic range. However, a solution for a continuous simply supported circular plate has been obtained by Moshaiov and Vorus, and offers a qualitative comparison with the results for the simply supported/guided

annular plate.

Plate deflection at the onset of yielding is shown in Figure 7-10 and at several loads in the plastic range in Figure 7-11 for the present analysis method. The deflection at yielding obtained using the analytic solution from Roark is shown in Figure 7-10 for comparison. Figure 7-12 shows the plastic zones in the plate at the three load conditions depicted in Figure 7-11 for the present solution. Corresponding results from Moshaiov and Vorus for the continuous plate of Figure 7-13 are shown in Figures 7-14 and 7-15.

$$\sigma_y = 16 \text{ KSI}$$

$$E = 10 \times 10^3 \text{ KSI}$$

$$\nu = 0.24$$

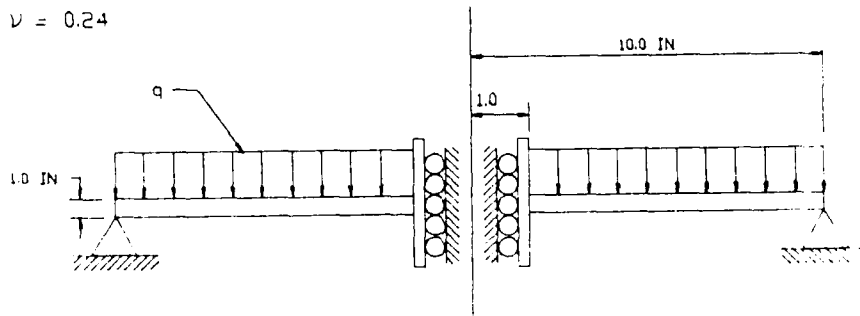


Figure 7-8. Simply supported, guided annular plate problem description.

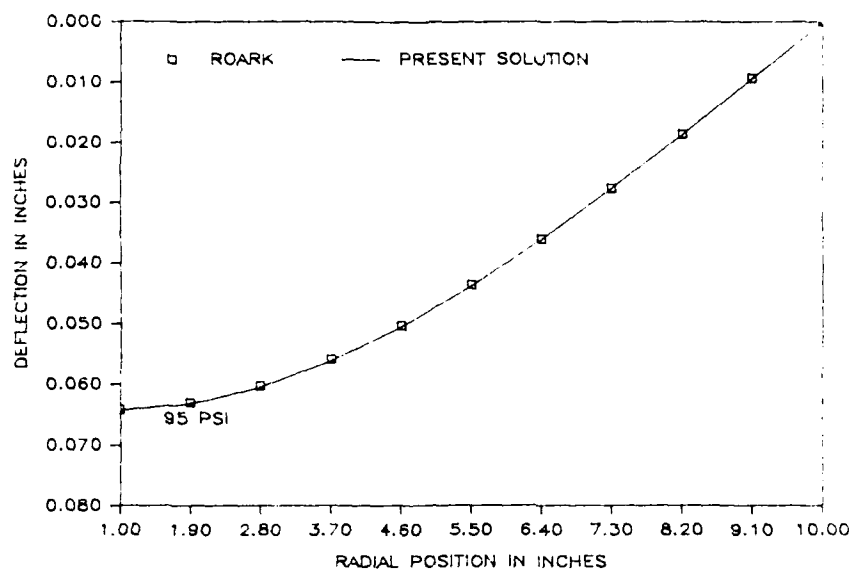


Figure 7-10. Deflection of simply supported guided annular plate at the onset of yielding

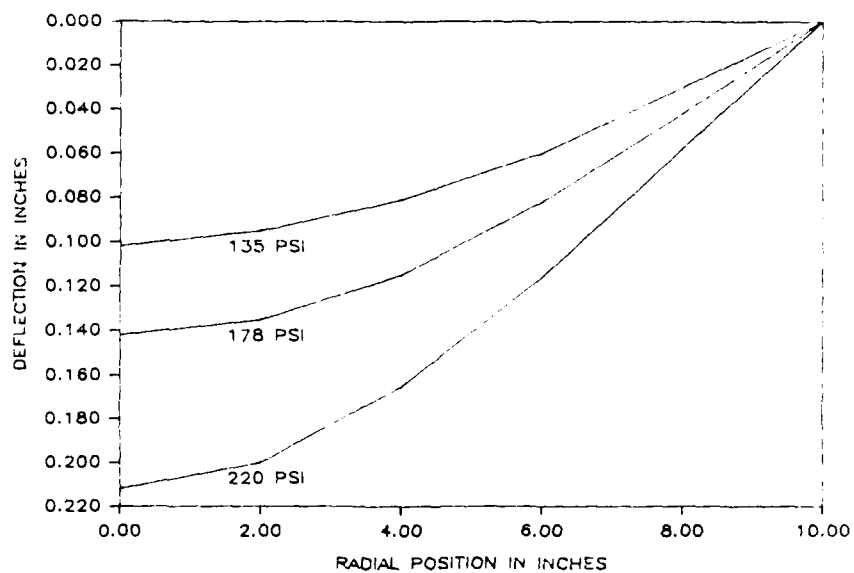


Figure 7-11. Deflection of simply supported guided annular plate in the plastic range

7.4 Discussion of Results

The results presented in Sections 7.1 through 7.3 for the elastic range are uniformly in excellent agreement with the results using the analytic solutions given by Roark. This is as should be expected, since the elastic range solution using the present method is based on a closed form analytic solution.

Results in the plastic range, while reasonable, are not in as close agreement with other published results. The only case where data for a direct comparison could be found is the configuration discussed in Section 7.1. From Figure 7-3, the deflections at loads of 652 psi and 852 psi for the present solution are in excellent agreement with the finite element results of Armen et al.

Results for the annular plate of Section 7.1 at the 1052 psi load are not as encouraging. Deflections and the extent of the plastic zone are greater for the solution obtained by Armen than for the present solution. To explain this difference, the size of the integration and load increments has been varied, but it shows little effect on the results. The reason behind this difference is not known. However, it is believed that it might be due to a programming error either in this report or in the reference. Also, differences could arise by not taking into account additional integration terms arising from the singularity at $r=r_0$, as discussed in Appendix A.

Finally, it is of interest to compare the results for the simply supported/guided plate of Figure 7-9 to the results for a simply-supported circular plate obtained by Moshaiov and Vorus. It is recognized that the two cases are not equivalent. However, one would expect the two plates to behave globally in a similar fashion. This is in fact the case, with both deflections and plastic zones showing reasonably good agreement in view of the differences between the two configurations.

CHAPTER 8

SUMMARY AND CONCLUSIONS

8.1 Summary

This thesis has developed a formulation for solving axisymmetrically loaded annular plate bending problems using boundary integrals. This particular formulation is unique in that it treats the annular plate as a one-dimensional problem, using "ring" type Green's functions to determine unknown boundary conditions and arrive at a plate bending solution. The formulation allows a numerical incremental load method to be used in the plastic range for the treatment of non-linear behavior. Results in the elastic range show excellent agreement with results obtained using a conventional analytic solution. In the plastic range, results are in reasonable agreement with those obtained using a finite element method.

8.2 Conclusions

This thesis treats the two-dimensional problem of analysis of axisymmetrically loaded annular plates as a one-dimensional problem. Using a method of solving one-dimensional problems

with boundary integrals developed by Butterfield, the thesis demonstrates that a closed form solution for the annular plate problem can be obtained. It further demonstrates that this approach is suitable for the use of an incremental load method to obtain a solution in the plastic range.

The formulation developed in this thesis is advantageous over a conventional analytic solution in that it allows a wide range of problems with different loading and boundary conditions to be solved using a single algorithm. Further, the simplicity of the approach is useful from an educational standpoint in the teaching of boundary element methods.

8.3 Recommendations for Future Work

There remains room for much additional work in this area. Some suggestions follow:

1. A formulation for axisymmetrically loaded continuous circular plates remains to be developed. The work presented here lays a foundation for the continuous plate formulation. However, extending this method to the continuous plate case may require some modifications.

2. In developing a formulation for the simple beam problem using boundary integrals, Butterfield uses both a direct and an indirect method, and shows that they yield the same result. Only the direct method is presented in this

thesis. The indirect method is a more intuitive approach than the direct method, and offers advantages in furthering the understanding of a boundary integral solution. It would, therefore, be worthwhile to pursue an indirect method as well as a direct method in developing a boundary integral formulation for circular plate geometries.

3. The Green's functions selected for this analysis involve many terms and are awkward from a computational standpoint. More thought needs to be given to selecting a "ring" type Green's function that is simpler and therefore offers advantages in simplifying calculations and reducing computational time.

4. For the plastic range, the lack of close agreement between the results presented in this work and those obtained by Armen et al using a finite element method need to be better understood. Additional comparative data should be found or developed to increase confidence in the results presented in this work.

APPENDIX A

SELECTED GREEN'S FUNCTIONS

This Appendix presents the selected Green's functions and required derivatives that are used to solve the plate bending problems in this analysis. The functions which mathematically describe the deflection, slope, radial and tangential moments, and shear are obtained from Roark.

A.1 Description of Geometry and Sign Conventions

As discussed in Chapter 4, the first Green's function selected for this analysis describes the response of annular plate that is simply supported at the outer radius and unsupported along the inner radius to a ring load. Figure A-1 depicts this plate configuration and important notation.

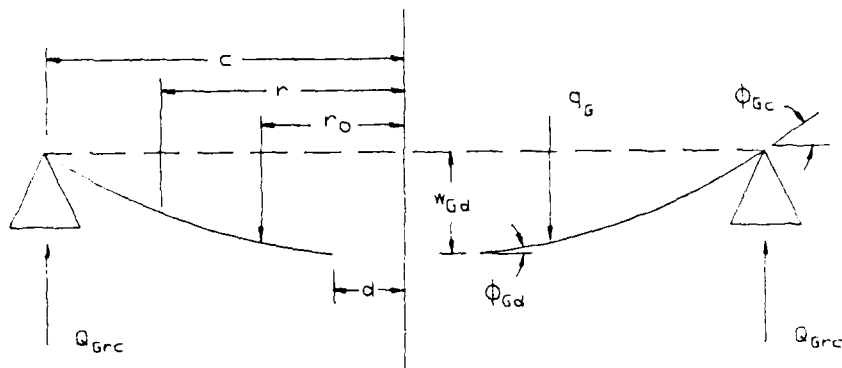


Figure A-1. Representation of Green's function used in the analysis.

Roark uses a sign convention with deflection w_G positive upward, slope ϕ_G positive when the deflection w_G increases positively as r increases, moment M_r positive when creating compression on the top surface of the plate, and the shear force Q_{Gr} positive when acting upward on the inner edge of an annular section. Subscripts c and d refer to the radial position.

This sign convention differs from the sign convention of Timoshenko, which has been adopted for this analysis. Accordingly, adjustments must be made. Specifically, Timoshenko takes deflection to be positive downward and shear to be positive when acting downward on the inner edge of the

plate, such that the signs for these items as given by Roark must be changed.

For slope, Roark uses a convention opposite to that of Eq. (2-2-1) and the convention used by Timoshenko. This difference, combined with the difference in defining the sign of deflection, effectively cancel each other, so the sign of slope from Roark does not change for this analysis. Finally, for moments, Roark and Timoshenko use the same sign convention, so no sign changes for moment terms are necessary.

The formulas in the sections which follow include a number of general plate functions and constants that are not depicted in Figure A-1. Functions L and G and their derivatives are included in Section A.7; constants F and C are included in Section A.8.

A.2 Singularities

It is not possible to evaluate the Green's function and associated derivatives at the exact radial location where the ring load is applied, due to singularities. To avoid this problem on the boundaries, the ring load is positioned a small distance inside the outer plate radius and outside the inner plate radius (0.00001 inches for the results presented in Chapter 7). During final review of this thesis, it was noted that because of the singularity at $r=r_0$, the domain integrand involving plastic moments of Eqs. (5-4-2) through (5-4-6)

includes singularities. Therefore, the limiting values should be explored and a correction applied in way of the singularities. Insufficient time was available to repeat the analysis with the applied correction to see its effect on the results. Note that this correction affects only the solution in the plastic range.

A.3 Deflection, Slope, Moment and Shear

The following expressions describe deflection w_G , slope $\dot{\phi}_G$, radial moment M_{Gr} , and shear Q_G corresponding to the first Green's function:

$$w_G = -w_{Gd} - \dot{\phi}_{Gd} r F_1 + \rho_G \frac{r^3}{D} G_3$$

$$\dot{\phi}_G = \dot{\phi}_{Gd} F_4 - \rho_G \frac{r^2}{D} G_6$$

$$M_{Gr} = \dot{\phi}_{Gd} \frac{D}{r} F_7 - \rho_G r G_9$$

$$Q_G = \rho_G \frac{r^0}{r} \langle r - r_0 \rangle^0$$

where

$$w_{Gd} = - \frac{\rho_G C^3}{D} \left(\frac{C_1 L_9}{C_7} - L_3 \right)$$

$$\ddagger_{Gd} = \frac{\rho_G C^2}{D C_7} L_9$$

D = plate flexural rigidity (Eq. (2-1-8))

ρ_G = magnitude of ring load

$\langle r-r_0 \rangle^0$ = singularity function

The expression $\langle r-r_0 \rangle^0$ represents a singularity function. The function is equal to zero if $r < r_0$. If $r > r_0$, the brackets become like any other brackets. Hence, for $r > r_0$, $\langle r-r_0 \rangle^0$ is $(r-r_0)$ to the power of 0, which equals to 1.

A.4 First Derivative of Deflection, Slope, Moment and Shear

The expressions which follow describe the first derivatives with respect to r_0 of w_G , \ddagger_G , M_{Gr} and Q_{Gr} . The first derivative of w_G corresponds to the second Green's function required by the analysis.

$$\frac{dw_G}{dr_0} = - \frac{dw_{Gd}}{dr_0} - \frac{d\ddagger_{Gd}}{dr_0} r F_1 + \rho_G \frac{r^3}{D} \frac{dG_3}{dr_0}$$

$$\frac{d\ddagger_G}{dr_0} = \frac{d\ddagger_{Gd}}{dr_0} F_4 - \rho_G \frac{r^2}{D} \frac{dG_6}{dr_0}$$

$$\frac{dM_{Gr}}{dr_0} = \frac{d\ddagger_{Gd}}{dr_0} \frac{D}{r} F_7 - \rho_G r \frac{dG_9}{dr_0}$$

$$\frac{dQ_{Gr}}{dr_o} = \frac{\rho_G}{r} \langle r - r_o \rangle^0$$

where

$$\frac{dw_{Gd}}{dr_o} = - \frac{\rho_G c^3}{D} \left(\frac{C_1 \frac{dL_9}{dr_o}}{C_7} - \frac{dL_3}{dr_o} \right)$$

$$\frac{d\ddot{w}_{Gd}}{dr_o} = \frac{\rho_G c^2}{D C_7} \frac{dL_9}{dr_o}$$

A.5 Second Derivative of Deflection, Slope, Moment and Shear

The following expressions describe the second derivatives with respect to r_o of w_G , \ddot{w}_G , M_{Gr} and Q_{Gr}

$$\frac{d^2 w_G}{dr_o^2} = - \frac{d^2 w_{Gd}}{dr_o^2} - \frac{d^2 \ddot{w}_{Gd}}{dr_o^2} r F_1 + \rho_G \frac{r^3}{D} \frac{d^2 G_3}{dr_o^2}$$

$$\frac{d^2 \ddot{w}_G}{dr_o^2} = \frac{d^2 \ddot{w}_{Gd}}{dr_o^2} F_4 - \rho_G \frac{r^2}{D} \frac{d^2 G_6}{dr_o^2}$$

$$\frac{d^2 M_{Gr}}{dr_o^2} = \frac{d^2 \ddot{w}_{Gd}}{dr_o^2} \frac{D}{r} F_7 - \rho_G r \frac{d^2 G_9}{dr_o^2}$$

$$\frac{d^2 Q_{Gr}}{dr_o^2} = 0$$

where

$$\frac{d^2 w_{Gd}}{dr_o^2} = - \frac{\rho_G C^3}{D} \left(\frac{C_1 \frac{d^2 L_3}{dr_o^2}}{C_7} - \frac{d^2 L_3}{dr_o^2} \right)$$

$$\frac{d^2 \phi_{Gd}}{dr_o^2} = \frac{\rho_G C^2}{D C_7} \frac{d^2 L_3}{dr_o^2}$$

A.6 Third Derivative of Deflection, Slope, Moment and Shear

The following expressions describe the third derivatives with respect to r_o of w_G , ϕ_G , M_{Gr} and Q_{Gr}

$$\frac{d^3 w_G}{dr_o^3} = - \frac{d^3 w_{Gd}}{dr_o^3} - \frac{d^3 \phi_{Gd}}{dr_o^3} r F_1 + \rho_G \frac{r^3}{D} \frac{d^3 G_3}{dr_o^3}$$

$$\frac{d^3 \phi_G}{dr_o^3} = \frac{d^3 \phi_{Gd}}{dr_o^3} F_4 + \rho_G \frac{r^2}{D} \frac{d^3 G_6}{dr_o^3}$$

$$\frac{d^3 M_{Gr}}{dr_o^3} = \frac{d^3 \phi_{Gd}}{dr_o^3} \frac{D}{r} F_7 - \rho_G r \frac{d^3 G_9}{dr_o^3}$$

$$\frac{d^3 Q_{Gr}}{dr_o^3} = 0$$

where

$$\frac{d^3 w_{Gd}}{dr_o^3} = - \frac{\rho_G c^3}{D} \left[\frac{C_1 \frac{d^3 L_9}{dr_o^3}}{C_7} - \frac{d^3 L_3}{dr_o^3} \right]$$

$$\frac{d^3 \phi_{Gd}}{dr_o^3} = \frac{\rho_G c^2}{D C_7} \frac{d^3 L_9}{dr_o^3}$$

A.7 Plate Functions L3, L9, G3, G6 and G9 and Their Derivatives

$$L_3 = \frac{r_o}{4c} \left\{ \left[\left(\frac{r_o^2}{c} \right) + 1 \right] \ln \frac{c}{r_o} + \left(\frac{r_o^2}{c} \right) - 1 \right\}$$

$$\frac{dL_3}{dr_o} = \frac{1}{4c} \left\{ \ln \frac{c}{r_o} \left[\frac{3r_o^2}{c^2} + 1 \right] + 2 \left[\frac{r_o^2}{c^2} - 1 \right] \right\}$$

$$\frac{d^2 L_3}{dr_o^2} = \frac{1}{4c} \left\{ \frac{6r_o}{c^2} \ln \frac{c}{r_o} + \frac{r_o}{c^2} - \frac{1}{r_o} \right\}$$

$$\frac{d^3 L_3}{dr_o^3} = \frac{1}{4c} \left\{ \frac{6}{c^2} \left[\ln \frac{c}{r_o} - 1 \right] + \frac{1}{c^2} + \frac{1}{r_o^2} \right\}$$

$$L_9 = \frac{r_o}{c} \left\{ \frac{1+\nu}{2} \ln \frac{c}{r_o} + \frac{1-\nu}{4} \left[1 - \left(\frac{r_o}{c} \right)^2 \right] \right\}$$

$$\frac{dL_9}{dr_0} = \frac{1}{4c} \left\{ 2(1+v) \ln \frac{c}{r_0} - (3v+1) - 3(1-v) \frac{r_0^2}{c^2} \right\}$$

$$\frac{d^2L_9}{dr_0^2} = -\frac{1}{2c} \left\{ \frac{1+v}{r_0} + \frac{3r_0(1-v)}{c^2} \right\}$$

$$\frac{d^3L_9}{dr_0^3} = \frac{1}{2c} \left\{ \frac{(1+v)}{r_0^2} - \frac{3(1-v)}{c^2} \right\}$$

$$G_3 = \frac{r_0}{4r} \left\{ \left[\left(\frac{r_0}{r} \right)^2 + 1 \right] \ln \frac{r}{r_0} + \left(\frac{r_0}{r} \right)^2 - 1 \right\} \langle r-r_0 \rangle^0$$

$$\frac{dG_3}{dr_0} = \frac{1}{4r} \left\{ \left[\frac{3r_0^2}{r^2} + 1 \right] \ln \frac{r}{r_0} + \frac{2r_0^2}{r^2} - 2 \right\} \langle r-r_0 \rangle^0$$

$$\frac{d^2G_3}{dr_0^2} = \frac{1}{4r} \left\{ \frac{6r_0}{r^2} \ln \frac{r}{r_0} + \frac{r_0}{r^2} - \frac{1}{r_0} \right\} \langle r-r_0 \rangle^0$$

$$\frac{d^3G_3}{dr_0^3} = \frac{1}{4r} \left\{ \frac{6}{r^2} \left[\ln \frac{r}{r_0} - 1 \right] + \frac{1}{r^2} + \frac{1}{r_0^2} \right\} \langle r-r_0 \rangle^0$$

$$G_6 = \frac{r_0}{4r} \left\{ \left(\frac{r_0}{r} \right)^2 - 1 + 2 \ln \frac{r}{r_0} \right\} \langle r-r_0 \rangle^0$$

$$\frac{dG_6}{dr_0} = \frac{1}{4r} \left\{ 3 \left(\frac{r_0^2}{r^2} - 1 \right) + 2 \ln \frac{r}{r_0} \right\} \langle r-r_0 \rangle^0$$

$$\frac{d^2 G_6}{dr_0^2} = \left\{ \frac{3r_0}{2r^3} - \frac{1}{2r_0 r} \right\} \langle r-r_0 \rangle^0$$

$$\frac{d^3 G_6}{dr_0^3} = \frac{1}{2r} \left\{ \frac{3}{r^2} + \frac{1}{r_0^2} \right\} \langle r-r_0 \rangle^0$$

$$G_9 = \frac{r_0}{r} \left\{ \frac{1+v}{2} \ln \frac{r}{r_0} + \frac{1-v}{4} \left[1 - \left(\frac{r_0}{r} \right)^2 \right] \right\} \langle r-r_0 \rangle^0$$

$$\frac{dG_9}{dr_0} = \frac{1}{4r} \left\{ 2(1+v) \left[\ln \frac{r}{r_0} - 1 \right] + (1-v) - \frac{3(1-v)r_0^2}{r^2} \right\} \langle r-r_0 \rangle^0$$

$$\frac{d^2 G_9}{dr_0^2} = -\frac{1}{4r} \left\{ \frac{2(1+v)}{r_0} + \frac{6r_0(1-v)}{r^2} \right\} \langle r-r_0 \rangle^0$$

$$\frac{d^3 G_9}{dr_0^3} = \frac{1}{4r} \left\{ \frac{2(1+v)}{r_0^2} - \frac{6}{r^2} (1-v) \right\} \langle r-r_0 \rangle^0$$

A.8 Plate Constants F_1 , F_4 , F_7 , C_1 , and C_7

$$F_1 = \frac{1 + \nu}{2} \frac{d}{r} \ln \frac{r}{d} + \frac{1 - \nu}{4} \left(\frac{r}{d} - \frac{d}{r} \right)$$

$$F_4 = \frac{1}{2} \left[(1 + \nu) \frac{d}{r} + (1 - \nu) \frac{r}{d} \right]$$

$$F_7 = \frac{1}{2} (1 - \nu^2) \left(\frac{r}{d} - \frac{d}{r} \right)$$

$$C_1 = \frac{1 + \nu}{2} \frac{d}{c} \ln \frac{c}{d} + \frac{1 - \nu}{4} \left(\frac{c}{d} - \frac{d}{c} \right)$$

$$C_7 = \frac{1}{2} (1 - \nu^2) \left(\frac{c}{d} - \frac{d}{c} \right)$$

A.9 Derivatives of the Green's Function With Respect to r

The equations describing non-linear plate behavior included in Section 5-4 are in terms of the first and second derivatives of the Green's function w_G with respect to r . These derivatives can be expressed with the aid of Eqs. (2-1-1) and (2-1-7) in terms of moments and slopes as follows:

$$\frac{dw_G}{dr} = -\phi_G$$

$$\frac{d^2 w_G}{dr^2} = -\frac{M_G r}{D} + \frac{\nu}{r} \phi_G$$

Expressions for slopes and moments for the Green's

function (Φ_G and M_{Gr}) are available and have been presented in Section A.3. It is further noted that Eqs. (5-4-2) through (5-4-6) require derivatives of the above two expressions with respect to r_0 . This is accomplished via the expressions for derivatives of Φ_G and M_{Gr} with respect to r_0 included in Sections A.5 through A.7 above.

APPENDIX B

COMPUTER PROGRAMS

This Appendix contains the computer programs developed to perform the analysis work. A flow chart showing the overall program structure is presented in Chapter 6.

The Appendix is organized with the programs INPUT and LOAD presented first, followed by the program MAIN. The supporting subroutines for the program MAIN follow that program.

```

C
C*
C*****
C*
C* PROGRAM INPUT
C*
C* THIS PROGRAM ALLOWS THE USER TO DEFINE THE PROBLEM OF INTEREST
C* AND TO SET CERTAIN ADJUSTABLE PARAMETERS SUCH AS NUMBER OF DEPTH
C* INTEGRATION INCREMENTS. LOADING CONDITIONS ARE INPUT USING THE
C* PROGRAM LOAD. TO RUN, THIS PROGRAM MUST ACCESS THE FILES GEOM.DAT,
C* BC.DAT, BCPOS.DAT AND ADJ.DAT
C*
C*****
C*
C
C PROGRAM INPUT
C
C IMPLICIT REAL*8 (A-H,O-Z)
C CHARACTER*1 BFLAG1(4),BFLAG2(4), RESP
C CHARACTER*19 LBL(8),LGEOM(9)
C CHARACTER*25 LADJ(6)
C
C REAL*8 ADJ(7),BC(4),GEOM(9),NU
C
C INTEGER POS(8),BCPOS(8),M,N
C
C*
C*****
C*
C* INPUT MATERIAL CONSTANTS AND PLATE GEOMETRY. THE FLEXURAL
C* RIGIDITY IS CALCULATED BASED ON THE INPUT PARAMETERS. N IS THE
C* TOTAL NUMBER OF PARAMETERS
C*
C*****
C*
C
C N=9
C
C LGEOM(1)='OUTER RADIUS A = '
C LGEOM(2)='INNER RADIUS B = '
C LGEOM(3)='POISSON RATIO NU = '
C LGEOM(4)='PLATE CONSTANT D = '
C LGEOM(5)='THICKNESS T = '
C LGEOM(6)='YIELD STRESS = '
C LGEOM(7)='YIELD STRAIN = '
C LGEOM(8)='ULTIMATE STRESS = '
C LGEOM(9)='ULTIMATE STRAIN = '
C
C WRITE(*,1)
1 FORMAT(//,6X,'WOULD YOU LIKE TO SEE THE CURRENT PLATE DIMENSIONS
//,2X,'AND MATERIAL CONSTANTS BY N = ',
READ(*,30) RESP
C
IF (RESP.EQ.'Y') THEN
WRITE(*,2

```



```

2      FORMAT(//,2X,'THE CURRENT VALUES FOLLOW. NOTE THAT A ZERO',
+        //,2X,'VALUE IS SHOWN FOR D WHEN T AND E ARE GIVEN AND VICE',
+        //,2X,'VERSA ...',/)
      OPEN(6,FILE='GEOM.DAT',STATUS='OLD')
      WRITE(*,30) ' '
      DO 8 I=1,N
        READ(6,70) GEOM(I)
        WRITE(*,6) LGEOM(I),GEOM(I)
6      FORMAT(2X,A22,F20.10)
8      CONTINUE
      CLOSE(6)
      ENDIF
C
      WRITE(*,9)
9      FORMAT(//,6X,'DO YOU WISH TO INPUT OR CHANGE GEOMETRY OR',
+        //,2X,'MATERIAL CONSTANTS (Y/N) ? ',\ )
      READ(*,30) RESP
C
      IF (RESP .EQ. 'Y') THEN
C
        WRITE(*,10)
        FORMAT(//,2X, 'INPUT THE VALUES USING DECIMAL NOTATION ....',/)
        DO 12 I=1,3
          WRITE(*,20) LGEOM(I)
          READ(*,70) GEOM(I)
12       CONTINUE
C
        DO 13 I=5,9
          WRITE(*,20) LGEOM(I)
          READ(*,70) GEOM(I)
13       CONTINUE
        GEOM(4)=GEOM(6)/GEOM(7)*GEOM(5)**3.0/12.0/(1.0-GEOM(3)**2.0)
C
        OPEN(6,FILE='GEOM.DAT',STATUS='NEW')
        DO 15 I=1,N
          WRITE(6,70) GEOM(I)
15       CONTINUE
        CLOSE(6)
      ENDIF
C
C
C *****
C INPUT PLATE BOUNDARY CONDITIONS
C *****
C
C
      LBL(1)='SHEAR AT A      = '
      LBL(2)='MOMENT AT A     = '
      LBL(3)='SLOPE AT A      = '
      LBL(4)='DEFLECTION AT A = '
      LBL(5)='SHEAR AT B      = '
      LBL(6)='MOMENT AT B     = '
      LBL(7)='SLOPE AT B      = '

```

```

      LBL(8)='DEFLECTION AT B = '
C
      OPEN(14,FILE='BCLBL.DAT',STATUS='NEW')
      DO 155 I=1,8
        WRITE(14,156) LBL(I)
155    CONTINUE
156    FORMAT(A22)
      CLOSE(14)
C
      WRITE(*,160)
160    FORMAT(//,6X,'WOULD YOU LIKE TO SEE THE CURRENT PLATE BOUNDARY'
+      ,//,2X,'CONDITIONS (Y/N) ? ',\))
      READ(*,30) RESP
C
      IF (RESP.EQ. 'Y') THEN
        OPEN(11,FILE='BC.DAT',STATUS='OLD')
        OPEN(12,FILE='BCPOS.DAT',STATUS='OLD')
        WRITE(*,30) ' '
        DO 165 I=1,4
          READ(11,70) BC(I)
165    CONTINUE
        DO 190 I=1,8
          READ(12,100) BCPOS(I)
          IF (BCPOS(I).GT. 0) THEN
            WRITE(*,186) LBL(I),BC(BCPOS(I))
186          FORMAT(2X,A22,F20.10)
          ENDIF
        CONTINUE
190    CLOSE(11)
        CLOSE(12)
      ENDIF
C
      WRITE(*,17)
17    FORMAT(//,6X,'DO YOU WISH TO INPUT OR CHANGE PLATE BOUNDARY',
+      ,//,2X,'CONDITIONS (Y/N) ? ',\))
      READ(*,30) RESP
C
      IF (RESP.EQ. 'Y') THEN
        WRITE(*,19)
19    FORMAT(//,6X,'INDICATE WHETHER THE BOUNDARY CONDITIONS FOR THE'
+      ,//,2X,'PLATE ARE KNOWN OR UNKNOWN BY TYPING IN THE LETTER "U"',
+      ,//,2X,'FOR UNKNOWN BOUNDARY CONDITIONS AND HITTING RETURN FOR',
+      ,//,2X,'KNOWN BOUNDARY CONDITIONS',//)
C
        WRITE(*,20) 'SHEAR AT A ? '
        READ(*,30) BFLAG1(1)
        WRITE(*,20) 'MOMENT AT A ? '
        READ(*,30) BFLAG1(2)
        WRITE(*,20) 'SLOPE AT A ? '
        READ(*,30) BFLAG1(3)
        WRITE(*,20) 'DEFLECTION AT A ? '
        READ(*,30) BFLAG1(4)
C
        WRITE(*,20) 'SHEAR AT B ? '
        READ(*,30) BFLAG2(1)

```

AD-A173 290

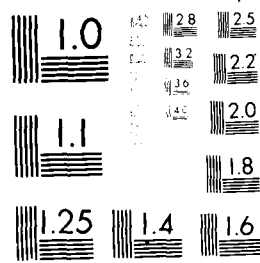
ELASTO-PLASTIC ANALYSIS OF AXISYMMETRICALLY LOADED
CIRCULAR PLATES USING... (U) MASSACHUSETTS INST OF TECH
CAMBRIDGE DEPT OF OCEAN ENGINEERING... P D EARECKSON
JUN 86 N00228-85-G-3262 F/G 20/11

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MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

```

WRITE(*,20) 'MOMENT AT B ? '
READ(*,30) BFLAG2(2)
WRITE(*,20) 'SLOPE AT B ? '
READ(*,30) BFLAG2(3)
WRITE(*,20) 'DEFLECTION AT B ? '
READ(*,30) BFLAG2(4)
20  FORMAT(2X,A22,\)
30  FORMAT(A1)
C
WRITE(*,40)
40  FORMAT(//,6X,'TYPE IN THE KNOWN BOUNDARY CONDITIONS USING ',
@2X,'DECIMAL NOTATION WHEN PROMPTED',//)
C
OPEN(8,FILE='BFLAG1.DAT',STATUS='NEW')
OPEN(9,FILE='BFLAG2.DAT',STATUS='NEW')
OPEN(10,FILE='BC.DAT',STATUS='NEW')
C
DO 50 I=1,4
WRITE(8,30) BFLAG1(I)
IF (BFLAG1(I) .NE. 'U') THEN
L=L+1
WRITE(*,20) LBL(I)
READ(*,70) BC(L)
WRITE(10,70) BC(L)
ENDIF
50  CONTINUE
C
DO 60 I=1,4
WRITE(9,30) BFLAG2(I)
IF (BFLAG2(I) .NE. 'U') THEN
L=L+1
WRITE(*,20) LBL(I+4)
READ(*,70) BC(L)
WRITE(10,70) BC(L)
ENDIF
60  CONTINUE
C
70  FORMAT(F20.10)
C
CLOSE(8)
CLOSE(9)
CLOSE(10)
C
OPEN(11,FILE='POS.DAT',STATUS='NEW')
OPEN(12,FILE='BCPOS.DAT',STATUS='NEW')
C
L=0
DO 80 I=1,4
IF (BFLAG1(I) .EQ. 'U') THEN
L=L+1
POS(I)=L
BCPOS(I)=0
ELSE
M=M+1
BCPOS(I)=M

```

```

      POS(I)=0
      ENDIF
C
      WRITE(11,100) POS(I)
      WRITE(12,100) BCPOS(I)
C
80  CONTINUE
C
      DO 90 I=5,8
      IF (/BFLAG2(I-4) .EQ. 'U') THEN
        L=L+1
        POS(I)=L
        BCPOS(I)=C
      ELSE
        M=M+1
        BCPOS(I)=M
        POS(I)=0
      ENDIF
C
      WRITE(11,100) POS(I)
      WRITE(12,100) BCPOS(I)
C
90  CONTINUE
      CLOSE(11)
      CLOSE(12)
100  FORMAT(11)
      ENDIF
C
C*
C*****
C*
C*   INPUT CALCULATION ADJUSTMENT ITEMS.  N IS THE NUMBER OF
C*   INPUT PARAMETERS IN THIS SUBMODULE.  OBSERVE THAT SUBSEQUENT TO
C*   THE DEVELOPMENT OF THIS MODULE, PARAMETER NUMBER 3 WAS NO LONGER
C*   ACCESSED FROM ANY OTHER PROGRAM, AND THEREFORE SERVES ONLY AS A
C*   DUMMY PARAMETER.  THE REMAINING PARAMETERS HAVE THE FOLLOWING
C*   MEANINGS
C*
C*   1.  EPSILON REFERS TO THE DISTANCE BETWEEN THE RING LOAD
C*   AND THE BOUNDARY.  NOTE THAT IN SOME INSTANCES, LETTING
C*   EPSILON EQUAL TO ZERO CAUSES A DIVIDE BY ZERO TYPE ERROR.
C*
C*   2.  THE INCREMENTS FOR R REFER TO THE STATIONS ACROSS
C*   THE PLATE.  THIS APPLIES TO THE NUMBER OF INTEGRATION
C*   INCREMENTS FOR DISTRIBUTED LOADS, AS WELL AS THE NUMBER
C*   OF DATA POINTS FOR THE RESULTS.
C*
C*   3.  THIS PARAMETER IS NO LONGER USED
C*
C*   4.  THIS REFERS TO THE EXTENT BY WHICH THE GREENS
C*   FUNCTION PLATE GEOMETRY OVERHANGS THE ACTUAL PLATE.
C*   FOR EXAMPLE, A VALUE OF 0.5 MEANS THAT THE INNER RADIUS OF
C*   THE RING LOADED PLATE DESCRIBED BY THE GREEN'S FUNCTION
C*   OVERHANGS THE ACTUAL PLATE BY 0.5 UNITS.
C*

```

```

C*      5. THE LOAD INCREMENT IS THE PERCENTAGE OF THE
C*      ORIGINALLY SPECIFIED LOAD MAGNITUDE THAT IS APPLIED DURING
C*      EACH LOAD INCREMENT AS THE PLATE IS LOADED IN THE PLASTIC
C*      RANGE
C*
C*      6. THIS IS THE NUMBER OF INTEGRATION INCREMENTS USED TO
C*      DETERMINE THE PLASTIC MOMENT ACROSS THE PLATE.
C*
C*      7. RMS IS THE ROOT MEAN SQUARE OF THE PLASTIC MOMENT
C*      INCREMENTS FOR EVERY POINT IN THE PLATE WHERE THE PLATE IS
C*      NO LONGER ELASTIC. THIS IS COMPARED WITH THE RMS VALUE FOR
C*      THE PREVIOUS INCREMENT, TO SEE WHETHER THE SOLUTION HAS
C*      SUFFICIENTLY CONVERGED. 1% CONVERGENCE MEANS THESE NUMBERS
C*      AGREE TO WITHIN LESS THAN 1%
C*
C*****
C*
C
      N=7
      LADJ(1)='EPSILON FOR RO = '
      LADJ(2)='INCREMENTS FOR R = '
      LADJ(3)='WIDTH INTG INCR = '
      LADJ(4)='BOUNDS FOR A AND B = '
      LADJ(5)='LOAD INCREMENT = '
      LADJ(6)='DEPTH INTG INCR = '
      LADJ(7)='% RMS CONVERGENCE = '
C
      WRITE(*,102)
102  FORMAT(//,6X,'WOULD YOU LIKE TO SEE THE CURRENT ADJUSTABLE'
+      //,2X,'PARAMETER SETTINGS' //)
      READ(*,30) RESP
      IF (RESP.EQ. 'Y') THEN
        OPEN(16,FILE='ADJ.DAT',STATUS='OLD')
        WRITE(*,30) ' '
        DO 108 I=1,N
          READ(16,70) ADJ(I)
          WRITE(*,106) LADJ(I),ADJ(I)
106  FORMAT(2X,A25,F20.10)
108  CONTINUE
        CLOSE(16)
      ENDIF
C
      WRITE(*,110)
110  FORMAT(//,6X,'DO YOU WISH TO INPUT OR CHANGE ANY COMPUTATIONAL'
+      //,2X,'ADJUSTMENT ITEMS (Y/N) ? ',\))
      READ(*,30) RESP
C
      IF (RESP.EQ. 'Y') THEN
C
        WRITE *,120
120  FORMAT //, ' INPUT THE VALUES USING DECIMAL NOTATION ....'
        DO 125 I=1,N
          WRITE *,126 LADJ(I)
          READ *,70 ADJ I
125  CONTINUE

```

```

126     FORMAT(2X,A25,\)
C
      OPEN(13,FILE='ADJ.DAT',STATUS='NEW')
      DO 130 I=1,N
        WRITE(13,70) ADJ(I)
130     CONTINUE
        CLOSE(13)
C
      ENDIF
C
      WRITE(*,135)
135     FORMAT(///,'  YOU ARE LEAVING THE INPUT MODULE',///)
C
      END

```



```

C
C*****
C*
C* PROGRAM LOAD
C*
C* THIS PROGRAM ALLOWS THE USER TO INPUT THE DESIRED LOADING
C* CONDITION FOR THE PLATE. UNIFORM DISTRIBUTED LOADS, RAMPS, AND
C* PARABOLICALLY DISTRIBUTED LOADS ARE PERMITTED. AS WELL AS RING
C* LOADS. TO RUN, THIS PROGRAM MUST ACCESS THE FILE LOAD.DAT.
C*
C* USING THE INPUT DATA, THE PROGRAM CALCULATES THE TOTAL LOAD
C* PER UNIT LENGTH THAT ACTS ON A RING OF THE PLATE SURFACE OF A
C* WIDTH DETERMINED BY ADJUSTABLE PARAMETER #3 OF THE INPUT
C* PROGRAM
C*
C*****
C*
C
C PROGRAM LOAD
C
C IMPLICIT REAL*8 (A-H,O-Z)
C CHARACTER*1 RESP
C CHARACTER*30 LBL(4)
C REAL*8 LD(50)
C INTEGER N
C
C LBL(1)='MAGNITUDE AT OUTER RADIUS = '
C LBL(2)='DISTANCE FROM PLATE CENTER = '
C LBL(3)='MAGNITUDE AT INNER RADIUS = '
C LBL(4)='DISTANCE FROM PLATE CENTER = '
C
C OPEN(6,FILE='LOAD.DAT',STATUS='OLD')
C READ(6,90) LD
C N=INT(6.0+2.0*LD(6))
C CLOSE(6)
C
C WRITE(*,10)
10 FORMAT(//,' DO YOU WISH TO SEE THE LOADING CONDITIONS',
+ /,' (Y/N) ? ',\ )
C READ(*,60) RESP
C IF (RESP.EQ. 'Y') THEN
C WRITE(*,15)
15 FORMAT(//,' THE EXISTING LOADING CONDITIONS FOLLOW ...',//)
C DO 40 I=1,4
C WRITE(*,30) LBL(I),LD(I)
30 FORMAT(2X,A30,F20.10)
40 CONTINUE
C WRITE(*,60)
C IF (LD(5).EQ. 1.0) THEN
C WRITE(*,80 'LOADING IS LINEAR'
C ELSE
C WRITE(*,80 'LOADING IS PARABOLIC'
C ENDIF
C WRITE(*,60)
C J=5

```

```

DO 45 I=7,6+INT(LD(6))
  J=J+2
  WRITE(*,60) ' '
  WRITE(*,43) 'MAGNITUDE OF RING LOAD',I-6, ' = ',LD(J)
  WRITE(*,43) 'LOCATION OF RING LOAD',I-6, ' = ',LD(J-1)
  FORMAT(2X,A24,I3,A3,F20.10)
43  CONTINUE
45  CONTINUE
ENDIF
C
WRITE(*,46)
46  FORMAT(//,2X,'DO YOU WISH TO CHANGE THE LOADING CONDITIONS',
+  /, ' (Y/N) ? ',\ )
READ(*,60) RESP
C
IF (RESP .EQ. 'Y') THEN
  WRITE(*,50)
50  FORMAT(//,2X,'DO YOU WISH TO CHANGE THE DISTRIBUTED LOAD',
+  /, ' (Y/N) ? ',\ )
  READ(*,60) RESP
60  FORMAT(A1)
C
IF (RESP .EQ. 'Y') THEN
C
  WRITE(*,70)
70  FORMAT(//,2X,'SPECIFY THE LOAD PROFILE USING DECIMAL',
+  /, ' NOTATION ... ',//)
  DO 75 I=1,4
    WRITE(*,80) LBL(I)
    READ(*,90) LD(I)
75  CONTINUE
80  FORMAT(2X,A30,\ )
90  FORMAT(F20.10)
C
  WRITE(*,100)
100  FORMAT(//,2X,'IS THE LOAD LINEAR OR PARABOLIC (L/P) ? ',\ )
  READ(*,60) RESP
  IF (RESP .EQ. 'L') THEN
    LD(5) = 1.0
  ELSE
    LD(5) = 2.0
  ENDIF
ENDIF
C
WRITE(*,110)
110  FORMAT(//,2X,'DO YOU WISH TO MODIFY ANY RING LOADS (Y/N) ? ',\ )
  READ(*,60) RESP
  IF (RESP .EQ. 'Y') THEN
    WRITE(*,120)
120  FORMAT(//,2X,'THE NUMBER OF RING LOADS (INTEGER 1-5)',\ )
    READ(*,130) N
130  FORMAT(I3)
    LD(6) = REAL(N)
    K=5
    DO 200 I=1,N
      K=K-2
    200  CONTINUE
  ENDIF

```

```

        WRITE *,60 ' '
        WRITE *,210 'MAGNITUDE OF RING LOAD',I, '= '
        READ *,90 LD,K
        WRITE *,210 'LOCATION OF RING LOAD ',I, '= '
        READ *,90 LD,K+1
200    CONTINUE
210    FORMAT(2X,A23,I3,A2,')
        ENDIF
C
        OPEN(7,FILE='LOAD.DAT',STATUS='NEW')
        DO 219 I=1,50
        WRITE(7,90) LD,I
219    CONTINUE
        ENDIF
C
        WRITE *,220
220    FORMAT(//,6X,'YOU ARE LEAVING THE LOADING MODULE',//)
        END

```

```

C
C*
C*****
C*
C*      CALL THE SUBROUTINE YLD, WHICH WILL RETURN A LOAD MATRIX
C*      SUCH THAT YIELDING HAS JUST STARTED TO OCCUR IN THE PLATE
C*
C*****
C*
C      CALL YLD(KK,IMAX,LDMAT1)
C
C*
C*****
C*      REPEAT THE PROCESS, AND THEN CALL UPDATE TO REVISE OUR STATE
C*      MATRIX TO REFLECT THE CONDITION OF THE PLATE AT THE ONSET OF
C*      YIELDING
C*
C*****
C*
C      CALL AMATR(BFLAG1,BFLAG2,BC,BCPOS,POS,LDMAT1,LHS,AMAT)
C      CALL GAUSS(LHS,AMAT,X)
C      CALL BCM(POS,BC,X,BCMAT)
C      CALL SOLVER(LDMAT1,BCMAT)
C      CALL UPDATE(STT1,IMAX,OUT,LDMAT1,LL,LDMATO)
C      CALL RESUL(M,STT1,COUNT,LDMATO)
C
C*
C*****
C*
C*      NOW, INPUT AN INCREMENTAL LOAD, AND COMPUTE THE RESULTING
C*      MOMENTS, DEFLECTIONS ETC, USING AS AN INPUT THE PLASTIC MOMENTS
C*      (IF ANY) AND OTHER DATA FROM THE PREVIOUS LOAD CASE
C*
C*****
C*
C
C      DO 110 I=1,NN
C          LDMAT1(I)=ADJ(5)*LDMATO(I)
C 110  CONTINUE
C
C 116  CONTINUE
C
C      WRITE(*,130) STT,IMAX,9J,STT(IMAX,10)
C 130  FORMAT(5X,'MRP IN = ',F20.10,' MTP IN = ',F20.10)
C
C      CALL AMATR BFLAG1,BFLAG2,BC,BCPOS,POS,LDMAT1,LHS,AMAT
C      CALL GAUSS LHS,AMAT,X
C      CALL BCM POS,BC,X,BCMAT
C      CALL SOLVER LDMAT1,BCMAT

```

```

C
C*
C*****
C*
C*   GO INTO THE SUBROUTINE PLAST TO CALCULATE THE INCREMENTAL PLASTIC
C*   MOMENTS AT EACH STATION ACROSS THE PLATE WHERE YIELDING HAS
C*   OCCURRED. THE PLASTIC MOMENTS THAT ARE RETURNED BY PLAST ARE
C*   SQUARED AND ADDED TO THE TOTAL IN THE VARIABLE RMS. RMS IS
C*   COMPARED TO THE RMS VALUE FROM THE PREVIOUS ITERATION (SAVED AS
C*   RMS1)
C*
C*****
C
C
C   RMS1=RMS
C   RMS=0.0
C   IZ=NINT(ADJ/6)
C   DO 200 I=1,NN
C     STT1(1,1)=STT(I,9)
C     STT1(1,2)=STT(I,10)
C     CALL PLAST(I)
C     IF (SET(I,IZ).GE. GEOM(6)) THEN
C       RMS=RMS+STT(I,9)**2.0+STT(I,10)**2.0
C     ENDIF
C 200 CONTINUE
C
C   WRITE *,810, RMS
C 810 FORMAT(5X,'RMS = ',F20.10)
C
C*
C*   THIS WAS THROWN IN TO AVOID A DIVIDE BY ZERO, UNTIL THE PROCESS
C*   HAS BEEN "PRIMED"
C*
C
C   IF (RMS.EQ. 0.0) THEN
C     RMS=1.0
C   ENDIF
C
C   IF (ABS(SQRT(RMS)-SQRT(RMS1))/SQRT(RMS)*100.0.GT. ADJ(7)) THEN
C     DO 815 I=1,NN
C       STT(I,9)=(STT(I,9)+STT1(I,1))/2.0
C       STT(I,10)=(STT(I,10)+STT1(I,2))/2.0
C 815 CONTINUE
C     GOTO 116
C   ENDIF

```

```

C
C*
C*****
C*
C*      CALL THE SUBROUTINE UPDATE, WHICH WILL UPDATE OUR STATE
C*      MATRIX AND OUR LOADING MATRIX TO REFLECT THE CONDITION OF THE
C*      PLATE AT THE END OF THE LOADING INCREMENT. THE PARAMETER
C*      COUNT KEEPS TRACK OF LOAD INCREMENTS FOR PURPOSES OF DISPLAYING
C*      RESULTS. WHEN COUNT=10, RESULTS ARE DISPLAYED AND WRITTEN TO
C*      FILE. IE EVERY 10 LOAD INCREMENTS.
C*
C*****
C*
C      CALL UPDATE(STT1,IMAX,OUT,LDMAT1,IL,LDMAT0
CALL RESUL:M,STT1,COUNT,LDMAT0
IF COUNT.EQ. 10.0 THEN
COUNT=0.0
ENDIF
COUNT=COUNT+1.0
M=M+1
WRITE(*,117) M
117 FORMAT(5X,'LOAD INCR = ',I3)
C
C*
C*****
C*
C*      CHECK TO SEE IF EXCEEDING ULTIMATE STRAIN. IF SO, KICK
C*      OUT OF THE PROCEDURE. IF NOT, WE SHOULD INCREMENT ON UP AGAIN
C*      BY USING AS OUR INPUT THE INCREMENTAL LOAD MATRIX.
C*
C*****
C*
C      IF (OUT.NE. 1) THEN
GOTO 116
ENDIF
C
C      CALL RESUL:M,STT1,COUNT,LDMAT0
C
C      CLOSE 15
C
C      END

```

```

C
C*
C*****
C*
C* SUBROUTINE INTEG
C*
C* THIS SUBROUTINE IS CALLED BY THE PROGRAM MAIN TO CREATE THE
C* ARRAY LDMAT. THIS ARRAY CONTAINS THE VALUES OF THE TOTAL LOAD
C* FOR EACH SEGMENT ALONG THE PLATE RADIUS. IT IS CONSTRUCTED
C* BY USING THE AVERAGE OF THE TWO ENDPOINTS FOR DISTRIBUTED
C* LOADS AND ADDING TO THIS VALUE THE MAGNITUDE OF ANY RING LOADS
C* THAT ARE SITUATED WITHIN THE SEGMENT.
C*****
C*
C
C SUBROUTINE INTEG(LD,LDMAT)
C
C IMPLICIT REAL*8 (A-H,O-Z)
C REAL*8 LDMAT(50),LD(50),NU
C INTEGER N
C $INCLUDE:'COMMON.FOR'
C
C A=GEOM(1)
C B=GEOM(2)
C NU=GEOM(3)
C D=GEOM(4)
C
C K=NINT(ADJ/2)
C DEL=(A-B)/ADJ/2
C R=B
C N=NINT(6.0-2.0*LD/6)
C
C
C*****
C*
C* CHECK TO SEE IF THERE IS A DISTRIBUTED LOAD. IF NOT, PROCEED
C* TO THE SECTION FOR RING LOADS
C*****
C*
C
C IF (LD(1).NE.0.0.OR.LD(3).NE.0.0) THEN
C
C R=B
C SLP=LD(1)+LD(3) LD(2)+LD(4)
C DO 100 I=1,K

```

```

C
C*
C*****
C*
C* SECTION FOR LINEARLY DISTRIBUTED LOADS
C*
C*****
C*
C      IF LD(5) .EQ. 1.0 THEN
C
C          R=R*DEL
C
C          IF (R-LD(2) .GT. DEL/10.0) THEN
C              LDMAT(1)=0.0
C
C          ELSE IF (R .GT. LD(4) THEN
C              LDMAT(1)=LP*(R-LD(4)-DEL/2.0-LD(3))*DEL
C
C          ELSE
C
C              LDMAT(1)=0.0
C
C          ENDDIF
C
C*****
C*
C* SECTION FOR PARABOLIC LOADS (Y=X**2)
C*
C*****
C*
C      ELSE IF (LD(5) .EQ. 2.0) THEN
C
C          R=R*DEL
C          P=(LD(2)-LD(4))*2.0/4.0/LD(1)-LD(3)
C          IF (LD(1) .GT. LD(3)) THEN
C              Y=(R-DEL/2.0-LD(4))*2.0/4.0/P-LD(3)
C          ELSE
C              Y=((LD(2)-R-DEL/2.0))*2.0/4.0/P-LD(4)
C          ENDDIF
C
C          IF (R-LD(2) .GT. DEL/10.0) THEN
C              LDMAT(1)=0.0
C
C          ELSE IF (R .GT. LD(4) THEN
C              LDMAT(1)=Y*DEL
C
C          ELSE
C
C              LDMAT(1)=0.0
C
C          ENDDIF
C      ENDDIF

```



```

C
100      CONTINUE
      ENDIF
C
C*
C*****
C*
C*      SECTION FOR ADDING IN RING LOADS
C*
C*****
C*
C
      J=5
      IF (N.NE.0) THEN
        DO 300 I=7,6-NINT(LD/6)
          J=J+2
          L=INT((LD(J+1)-B)/DEL)+1
          LDMAT(L)=LDMAT(L)+LD(J)
300      CONTINUE
      ENDIF
C
C
      RETURN
      END

```

```

C
C*
C*****
C*
C*      SUBROUTINE AMATR
C*
C*      THIS SUBROUTINE IS CALLED BY THE PROGRAM MAIN TO CREATE THE
C*      4X4 A MATRIX, WHICH IS SOLVED USING GAUSS ELIMINATION TO
C*      DETERMINE THE UNKNOWN BOUNDARY CONDITIONS.
C*****
C*
C
      SUBROUTINE AMATR(BFLAG1,BFLAG2,BC,BCPOS,POS,LDMAT1,LHS,AMAT)
C
      IMPLICIT REAL*8 A-H,O-Z
      CHARACTER*1 BFLAG1 4 , BFLAG2 4
      REAL*8 BC 4 , LHS 4 , INC,NU,GRNMT1 4,6 ,GRNMT2 4,6 ,
      *      AMAT 4,4 , LDMAT 4 , LDTOT 4 , LDMAT1 50 , MFRTOT 4 , MPTTOT 4 ,
      *      STAV 50,2
      INTEGER POS 8 , BCPOS 8 , M,N
C
$INCLUDE 'COMMON.FOR'
C
      A=GEOM 1

```

```

      B=GEOM(2;
      NU=GEOM(3;
      D=GEOM(4;
      INC=ADJ(1;
C
      DO 101 I=1,4
      LHS(I)=0.0
      IF (I.EQ. 1 .OR. I.EQ. 3) THEN
        RO=A-INC
      ELSE
        RO=B-INC
      ENDIF
C
      R=A
      CALL GRNFNC(R,RO,NU,D,GRNMT1;
      R=B
      CALL GRNFNC(R,RO,NU,D,GRNMT2;
C
C*
C*****
C*
C*   REDUCE THE TERMS ORIGINALLY ON THE RHS TO A 4X4 MATRIX
C*   COEFFICIENTS (AMAT) AND A 4X1 MATRIX OF NON-COEFFICIENT
C*   TERMS (LHS)
C*
C*****
C*
C
      IF (I.EQ. 1 .OR. I.EQ. 2) THEN
        K=0
        L=0
        DO 90 J=1,4
          IF (BFLAG1(J).NE. 'U') THEN
            K=K+1
            LHS(I)=GRNMT1(J,1)*BC(K)*A*(-1.0 ** J-1)-LHS(I)
          ELSE
            L=L+1
            AMAT(I,L)=GRNMT1(J,1)*A*(-1.0 ** J)
          ENDIF
        ENDIF
        FORMAT(2X,F10.5)
        CONTINUE
73
90
C
      DO 100 J=1,4
      IF (BFLAG2(J).NE. 'U') THEN
        K=K+1
        LHS 1 =GRNMT2(J,1)*BC(K)*B*(-1 ** J-1)
      ELSE
        L=L+1
        AMAT 1,L =GRNMT2(J,1)*B*(-1 ** J-1)
      ENDIF
100
      CONTINUE
C
C
      ELSE
        K=0

```

```

L=0
DO 110 J=1,4
  IF (BFLAG1(J) .NE. 'U') THEN
    K=K+1
    LHS(I)=GRNMT1(J,2)*BC(K)*A*(-1.0)**(J-1)-LHS(I)
  ELSE
    L=L+1
    AMAT(I,L)=GRNMT1(J,2)*A*(-1.0)**(J)
  ENDIF
CONTINUE
110
C
DO 115 J=1,4
  IF (BFLAG2(J) .NE. 'U') THEN
    K=K+1
    LHS(I)=GRNMT2(J,2)*BC(K)*B*(-1)**(J)+LHS(I)
  ELSE
    L=L+1
    AMAT(I,L)=GRNMT2(J,2)*B*(-1)**(J-1)
  ENDIF
CONTINUE
115
ENDIF
C
C#
C*****
C#
C#  MOVE THE TERMS ORIGINALLY ON THE LHS TO THE RHS, FORM A NEW AMAT
C#  MATRIX AND A NEW LHS MATRIX WHICH CORRESPOND TO THE A AND B MATRICES
C#  OF THE SYSTEM AX=B
C#
C*****
C#
  IF (BFLAG1(4) .NE. 'U' .AND. I .EQ. 1) THEN
    LHS(1)=LHS(1)-RO*BC(BCPOS(4))
  ELSE IF (BFLAG1(4) .EQ. 'U' .AND. I .EQ. 1) THEN
    AMAT(I,POS(4))=AMAT(I,POS(4))-RO
  ENDIF
C
  IF (BFLAG2(4) .NE. 'U' .AND. I .EQ. 2) THEN
    LHS(1)=LHS(2)+RO*BC(BCPOS(8))
  ELSE IF (BFLAG2(4) .EQ. 'U' .AND. I .EQ. 2) THEN
    AMAT(I,POS(8))=AMAT(I,POS(8))+RO
  ENDIF
C
  IF (BFLAG1(3) .NE. 'U' .AND. I .EQ. 3) THEN
    LHS(3)=LHS(3)-RO*BC(BCPOS(3))
  ENDIF
  IF (BFLAG1(4) .NE. 'U' .AND. I .EQ. 3) THEN
    LHS 3 =LHS 3 -BC BCPOS 4
  ENDIF
  IF (BFLAG1(3) .EQ. 'U' .AND. I .EQ. 3) THEN
    AMAT I,POS,3 =AMAT I,POS,3 +RO
  ENDIF
  IF (BFLAG1(4) .EQ. 'U' .AND. I .EQ. 3) THEN
    AMAT I,POS 4 =AMAT I,POS 4 -1.0
  ENDIF

```

```

C
  IF (BFLAG2(3) .NE. 'U' .AND. I .EQ. 4) THEN
    LHS(4)=LHS(4)-RO*BC(BCPOS(7))
  ENDIF
  IF (BFLAG2(4) .NE. 'U' .AND. I .EQ. 4) THEN
    LHS(4)=LHS(4)+BC(BCPOS(8))
  ENDIF
  IF (BFLAG2(3) .EQ. 'U' .AND. I .EQ. 4) THEN
    AMAT(I,POS(7))=AMAT(I,POS(7))+RO
  ENDIF
  IF (BFLAG2(4) .EQ. 'U' .AND. I .EQ. 4) THEN
    AMAT(I,POS(8))=AMAT(I,POS(8))-1.0
  ENDIF
C
101 CONTINUE
C
C*
C*****
C*   FINALLY, ADJUST THE LHS TO ACCOUNT FOR THE EXTERNAL LOAD
C*   AND PLASTICITY BY CALCULATING AND SUBTRACTING APPROPRIATE
C*   TERMS.
C*
C*****
C
K=NINT(ADJ(2))
DEL=(A-B./ADJ(2))
C
DO 500 I=1,K
  R=B-DEL/2.0+REAL(I)*DEL
  RO=A-ADJ(1)
  STTAV(1,1)=(STT(I,9)+STT(I+1,9))/2.0
  STTAV(1,2)=(STT(I,10)+STT(I+1,10))/2.0
  CALL GRNFNC(R,RO,NU,D,GRNMT1)
  LDTOT(1)=LDTOT(1)+LDMAT1(I)*GRNMT1(1,1)*R
  MPRTOT(1)=MPRTOT(1)+STTAV(1,1)*(-GRNMT1(3,1)/D+
    NU/R*GRNMT1(2,1))*DEL*R
  MPTTOT(1)=MPTTOT(1)+STTAV(1,2)*(-GRNMT1(2,1)*DEL
  LDTOT(3)=LDTOT(3)+LDMAT1(I)*GRNMT1(1,2)*R
  MPRTOT(3)=MPRTOT(3)+STTAV(1,1)*(-GRNMT1(3,2)/D+
    NU/R*GRNMT1(2,2))*DEL*R
  MPTTOT(3)=MPTTOT(3)+STTAV(1,2)*(-GRNMT1(2,2)*DEL)
  RO=B+ADJ(1)
  CALL GRNFNC(R,RO,NU,D,GRNMT1)
  LDTOT(2)=LDTOT(2)+LDMAT1(I)*GRNMT1(1,1)*R
  MPRTOT(2)=MPRTOT(2)+STTAV(1,1)*(-GRNMT1(3,1)/D+
    NU/R*GRNMT1(2,1))*DEL*R
  MPTTOT(2)=MPTTOT(2)+STTAV(1,2)*(-GRNMT1(2,1)*DEL
  LDTOT(4)=LDTOT(4)+LDMAT1(I)*GRNMT1(1,2)*R
  MPRTOT(4)=MPRTOT(4)+STTAV(1,1)*(-GRNMT1(3,2)/D+
    NU/R*GRNMT1(2,2))*DEL*R
  MPTTOT(4)=MPTTOT(4)+STTAV(1,2)*(-GRNMT1(2,2)*DEL)
500 CONTINUE
C

```

```

      DO 105 I=1,4
        LHS(I)=LHS(I)-LDTOT(I)+MPRTOT(I)+MPTTOT(I)
105  CONTINUE
C
      DO 1000 I=1,4
        LDTOT(I)=0.0
        MPRTOT(I)=0.0
        MPTTOT(I)=0.0
1000 CONTINUE
C
      RETURN
      END

```

```

C
C*
C*****
C*
C* SUBROUTINE GRNFNC
C*
C* THIS SUBROUTINE CALCULATES THE VALUES OF DEFLECTION, SLOPE,
C* MOMENT AND SHEAR AND ASSOCIATED DERIVATIVES FOR THE GREEN'S
C* FUNCTION, AND RETURNS THESE VALUES TO THE MAIN PROGRAM. TO
C* INDICATE DERIVATIVES OF PARAMETERS, THE NUMBER 1, 2, OR 3 IS
C* TACKED ON TO THE END OF THE VARIABLE NAME. HENCE, THE FIRST
C* DERIVATIVE OF L9 WRT PO IS L91
C*
C*
C*****
C*
C* SUBROUTINE GRNFNC(R,RO,NU,D,GRNMAT)
C
      IMPLICIT REAL*8 (A-R,O-Z)
      REAL*8 NU,GRNMAT(4,6),L3,L31,L32,L33,L9,L91,L92,L93
      INCLUDE 'COMMON.FOR'
C
      W=1.0
      A=GEOM(1)+ADJ(4)
      B=GEOM(2)-ADJ(4)
      THB=0.0
      MB=0.0
      QB=0.0
C
      IF (R .GT. RO) THEN
        RRO=1.0
      ELSE
        RRO=0.0
      ENDIF

```

```

C
C*
C*****
C*
C*   CALCULATE PLATE CONSTANTS AND PLATE FUNCTIONS, AND THEIR
C*   DERIVATIVES
C*
C*****
C*
C
  L3=RO/4.0/A*((RO/A)**2.0+1.0)*DLOG(A/RO)+(RO/A)**2.0-1.0)
  L31=1.0/4.0/A*(DLOG(A/RO)*(3.0*RO**2.0/A**2.0+1.0)+2.0*
  - (RO**2.0/A**2.0-1.0))
  L32=1.0/4.0/A*(6.0*RO/A**2.0*DLOG(A/RO)-RO/A**2.0-1.0/RO)
  L33=1.0/4.0/A*(6.0/A**2.0*(DLOG(A/RO)-1.0)+1.0/A**2.0+1.0/RO
  - **2.0)
C
  L9=RO/A*((1.0+NU)/2.0*DLOG(A/RO)+(1.0-NU)/4.0*(1.0-(RO/A)**2.0))
  L91=1.0/4.0/A*(2.0*(1.0+NU)*DLOG(A/RO)-(3.0*NU+1.0)-3.0*(1.0-NU)
  + *RO**2.0/A**2.0)
  L92=-(1.0-NU)/2.0/A/RO-3.0*RO*(1.0-NU)/2.0/A**3.0
  L93=(1.0+NU)/2.0/A/RO**2.0-3.0*(1.0-NU)/2.0/A**3.0
C
  C1=(1.0+NU)/2.0*B/A*DLOG(A/B)+(1.0-NU)/4.0*(A/B-B/A)
  C7=1.0/2.0*(1.0-NU**2.0)*(A/B-B/A)
C
  G3=RO/4.0/R*((RO/R)**2.0+1.0)*DLOG(R/RO)+(RO/R)**2.0-1.0)*RRO
  G31=1.0/4.0/R*(3.0*RO**2.0/R**2.0*DLOG(R/RO)+2.0*RO**2.0/R**2.0-
  - DLOG(R/RO)-2.0)*RRO
  G32=1.0/4.0/R*(6.0*RO/R**2.0*DLOG(R/RO)+RO/R**2.0-1.0/RO)*RRO
  G33=1.0/4.0/R*(6.0/R**2.0*(DLOG(R/RO)-1.0)+1.0/R**2.0+1.0/RO**2.0
  - *RRO)
C
  G6=RO/4.0/R*((RO/R)**2.0-1.0+2.0*DLOG(R/RO))*RRO
  G61=1.0/4.0/R*(3.0*RO**2.0/R**2.0-1.0)-2.0*DLOG(R/RO)*RRO
  G62=(3.0*RO/2.0/R**3.0-1.0/2.0/RO/R)*RRO
  G63=(3.0/2.0/R**3.0+1.0/2.0/RO/R**2.0)*RRO
C
  G9=RO/R*((1.0+NU)/2.0*DLOG(R/RO)+(1.0-NU)/4.0*(1.0-(RO/R)**2.0))
  + *RRO
  G91=1.0/4.0/R*(2.0*(1.0+NU)*DLOG(R/RO)-1.0)+(1.0-NU)-3.0*
  + (1.0-NU)*RO**2.0/R**2.0)*RRO
  G92=-1.0/4.0/R*(2.0*(1.0+NU)/RO+6.0*RO/R**2.0*(1.0-NU))*RRO
  G93=1.0/4.0/R*(2.0*(1.0+NU)/RO**2.0-6.0/R**2.0*(1.0-NU))*RRO
C
  F1=(1.0+NU)/2.0*B/R*DLOG(R/B)+(1.0-NU)/4.0*(R/B-B/R)
  F2=1.0/4.0*(1.0-B/R**2.0+1.0+2.0*DLOG(R/B))
  F3=B/4.0/R*(B/R**2.0-1.0)*DLOG(R/B)+B/R**2.0-1.0)
  F4=1.0/2.0*(1.0-NU)*B/R-1.0-NU)*R/B
  F5=1.0/2.0*(1.0-B/R**2.0
  F6=B/4.0/R*(B/R**2.0-1.0+2.0*DLOG(R/B))
  F7=1.0/2.0*(1.0-NU**2.0)*R/B-B/R
  F8=1.0/2.0*(1.0-NU*(1.0-NU)*B/R**2.0
  F9=B/R*(1.0-NU**2.0*DLOG(R/B)+(1.0-NU)/4.0*(1.0-B/R**2.0

```

```

C
C*
C*****
C*
C*   CALCULATE THE DEFLECTION, SLOPE, MOMENT AND SHEAR, AND
C*   ASSOCIATED DERIVATIVES, AND STORE IN THE MATRIX GRNMAT.
C*   THIS MATRIX IS A 4 BY 4 MATRIX, WHERE ROW POSITIONS ARE AS
C*   FOLLOWS
C*
C*       1   DEFLECTION
C*       2   SLOPE
C*       3   MOMENT
C*       4   SHEAR
C*
C*   AND COLUMN POSITIONS ARE AS FOLLOWS
C*
C*       1   GREEN'S FUNCTION (NO DERIVATIVES)
C*       2   1ST DERIVATIVE OF GREEN'S FUNCTION WRT RO
C*       3   2ND DERIVATIVE OF GREEN'S FUNCTION WRT RO
C*       4   3RD DERIVATIVE OF GREEN'S FUNCTION WRT RO
C*
C*   FOR EXAMPLE, THE VALUE RETURNED BY GRNMAT(2,3) REPRESENTS
C*   THE SECOND DERIVATIVE OF THE SLOPE FOR THE GREEN'S FUNCTION
C*   WRT RO
C*****
C*
C
C       YB=-W*A**3.0/D*(C1*L9/C7-L3)
C       THB=W*A**2.0/D C7*L9
C       GRNMAT(1,1)=-(YB-THB*R*F1-W*R**3.0/D*G3
C       GRNMAT(2,1)=THB*F4-W*R**2.0/D*G6
C       GRNMAT(3,1)=THB*D/R*F7-W*R*G9
C       GRNMAT(4,1)=W*RO/R*RRO
C
C
C       YB1=-W*A**3.0/D*(C1*L91/C7-L31)
C       THB1=W*A**2.0/D/C7*L91
C       GRNMAT(1,2)=-(YB1-THB1*R*F1-W*R**3.0/D*G31)
C       GRNMAT(2,2)=THB1*F4-W*R**2.0/D*G61
C       GRNMAT(3,2)=THB1*D/R*F7-W*R*G91
C       GRNMAT(4,2)=W/R*RRO
C
C
C       YB2=-W*A**3.0/D*(C1/C7*L92-L32)
C       THB2=W*A**2.0/D/C7*L92
C       GRNMAT(1,3)=-(YB2-R*F1*THB2-W*R**3.0/D*G32
C       GRNMAT(2,3)=F4*THB2-W*R**2.0/D*G62
C       GRNMAT(3,3)=D*F7/R*THB2-W*R*G92
C       GRNMAT(4,3)=0
C
C
C       YB3=-W*A**3.0/D*(C1/C7*L93-L33)
C       THB3=W*A**2.0/D/C7*L93
C       GRNMAT(1,4)=-(YB3-R*F1*THB3-W*R**3.0/D*G33

```

```

GRNMAT(2,4)=F4*TB3-W*R**2.0/D*G63
GRNMAT(3,4)=D*F7/R*TB3-W*R*G93
GRNMAT(4,4)=0

```

```

C      RETURN
      END

```

```

C
C*
C*****
C*
C*   SUBROUTINE GAUSS
C*
C*   THIS SUBROUTINE USES GAUSS ELIMINATION TO DETERMINE THE
C*   UNKNOWN BOUNDARY CONDITIONS. THE MATRICES A AND B OF THE
C*   EQUATION AX=B ARE PROVIDED BY THE SUBROUTINE AMATR. THIS
C*   SUBROUTINE THEN RETURNS THE MATRIX X OF UNKNOWN BOUNDARY
C*   CONDITIONS. THIS SUBROUTINE WAS TAKEN FROM THE BOOK
C*   "NUMERICAL ANALYSIS" BY L.W. JOHNSON AND R.D. RIESS.
C*
C*****
C*
C      SUBROUTINE GAUSS(B,A,X)
C
C      IMPLICIT REAL*8 (A-H,O-Z)
C      CHARACTER*22 BCLBL(8)
C      REAL*8 A(4,4),B(4),X(4),BCMAT(8),BC(4)
C      DIMENSION AUG(50,51)
C      INTEGER POS(8),BCPOS(8),M,N
C
C      NM1=3
C      NP=5
C      N=4
C
C      SET UP THE AUGMENTED MATRIX FOR AX=B
C
C      DO 2 I=1,N
C        DO 1 J=1,N
C          AUG(I,J)=A(I,J)
C1       CONTINUE
C        AUG(I,NP)=B(I)
C2       CONTINUE
C
C      THE OUTER LOOP USES ELEMENTARY ROW OPERATIONS TO TRANSFORM
C      THE AUGMENTED MATRIX TO ECHELON FORM

```



```

C      DO 8 I=1,NM1
C
C      SEARCH FOR THE LARGEST ENTRY IN COLUMN I, ROWS I THROUGH N
C      IPIVOT IS THE ROW INDEX OF THE LARGEST ENTRY
C
      PIVOT=0.0
      DO 3 J=I,N
        TEMP=ABS(AUG(J,I))
        IF(PIVOT.GE.TEMP) GO TO 3
        PIVOT=TEMP
        IPIVOT=J
      3    CONTINUE
      IF(PIVOT.EQ.0.0) GO TO 13
      IF(IPIVOT.EQ.I) GO TO 5
C
C      INTERCHANGE ROW I AND ROW IPIVOT
C
      DO 4 K=I,NP
        TEMP=AUG(I,K)
        AUG(I,K)=AUG(IPIVOT,K)
        AUG(IPIVOT,K)=TEMP
      4    CONTINUE
C
C      ZERO ENTRIES (I+1),(I+2),...,(N,I) IN THE AUGMENTED MATRIX
C
      IPI=I+1
      DO 7 K=IPI,N
        Q=-AUG(K,I)/AUG(I,I)
        AUG(K,I)=0.0
        DO 6 J=IPI,NP
          AUG(K,J)=Q*AUG(I,J)+AUG(K,J)
        6    CONTINUE
      7    CONTINUE
      8    CONTINUE
      IF(AUG(N,N).EQ.0.0) GO TO 13
C
C      BACKSOLVE TO OBTAIN A SOLUTION TO AX=B
C
      X(N)=AUG(N,NP)/AUG(N,N)
      DO 10 K=1,NM1
        Q=0.0
        DO 9 J=1,K
          Q=Q+AUG(N-K,NP-J)*X(NP-J)
        9    CONTINUE
        X(N-K)=AUG(N-K,NP)-Q/AUG(N-K,N-K)
      10   CONTINUE
C
C      CALCULATE THE NORM OF THE RESIDUAL VECTOR, B-AX.
C      SET IEHRR=1 AND RETURN
C
      RSQ=0.0
      DO 12 I=1,N
        Q=0.0
        DO 11 J=1,N

```

```

      Q=Q+A(I,J)*X(J)
11      CONTINUE
      RESI=B(I)-Q
      RMAG=ABS(RESI)
      RSQ=RSQ+RMAG**2
12      CONTINUE
      RNORM=SQRT(RSQ)
      IERROR=1
C
C   ABNORMAL RETURN --- REDUCTION TO ECHELON FORM PRODUCES A ZERO
C   ENTRY ON THE DIAGONAL. THE MATRIX A MAY BE SINGULAR.
C
13      IERROR=2
C
      RETURN
      END

```

```

C
C*
C*****
C*
C*   SUBROUTINE BCM
C*
C*   THIS SUBROUTINE TAKES THE KNOWN BOUNDARY CONDITIONS STORED
C*   BY THE PROGRAM INPUT IN THE FILE BCM.DAT AS WELL AS THE
C*   "UNKNOWN" BOUNDARY CONDITIONS RETURNED BY THE SUBROUTINE GAUSS
C*   AND CREATES THE 8X1 MATRIX OF BOUNDARY CONDITIONS BCMAT.
C*
C*****
C*
C   SUBROUTINE BCM(POS,BC,X,BCMAT)
C
C   IMPLICIT REAL*8 (A-H,O-Z)
C   REAL*8 BC(4),X(4),BCMAT(8)
C   INTEGER POS(8)
C
C
C   K=0
C   L=0
C   DO 120 I=1,8
C     IF (POS(I) .GT. 0) THEN
C       K=K+1
C       BCMAT(K)=X(I)
C     ELSE
C       L=L+1
C       BCMAT(L)=BC(I)
C     ENDIF
C   120 CONTINUE
C   RETURN
C   END

```

```

C
C*
C*****
C*
C* SUBROUTINE SOLVER
C*
C* THIS SUBROUTINE SOLVES TAKES THE COMPLETE SET OF BOUNDARY
C* CONDITIONS SUPPLIED BY THE SUBROUTINE BCM AND SOLVES THE PLATE
C* BENDING PROBLEM ACROSS THE PLATE WIDTH. IMPORTANT PARAMETERS
C* SUCH AS DEFLECTION, SLOPE ETC ARE STORED IN THE ARRAY ST
C*
C*****
C*
C SUBROUTINE SOLVER,XMT,BCMAT
C
C IMPLICIT REAL*8 (A-H,O-Z)
C
C REAL*8 GRNMT1(4,6),GRNMT2(4,6),BCMAT(8),DT(50,4),SM(50,4),NM
C * XMT(50),STTAV(50,2)
C INTEGER M,N
$INCLUDE:'COMMON.FOR'
C
C*
C*****
C* REZERO THE MATRICES DT AND ST
C*
C*****
C*
C
C DO 10 I=1,50
C   DO 5 J=1,4
C     SM(I,J)=0.0
C     DT(I,J)=0.0
C   5 CONTINUE
10 CONTINUE
C
C A=GEOM(1)
C B=GEOM(2)
C NU=GEOM(3)
C D=GEOM(4)
C
C
C RO=B
C DEL A B ADJ 2
C N INT ADJ 2

```

```

C
C*
C*****
C*
C*   THIS LOOP CALCULATES THE LOAD TERM AND THE PLASTIC MOMENT
C*   TERM FOR EACH EQUATION. I REFERS TO THE RADIAL STATION AND J
C*   REFERS TO THE EQUATION, WHERE THE FIRST EQUATION IS FOR
C*   DEFLECTION, THE SECOND FOR DW/DR, THE THIRD FOR D^2W/DR^2 AND
C*   THE FOURTH FOR D^3W/DR^3. THE DERIVATIVES ARE THEN USED TO
C*   CALCULATE SLOPES, MOMENTS AND SHEARS
C*
C*****
C*
C
C   DO 155 I=1,N+1
C
C       IF (I.EQ. 1). THEN
C           RO=B+ADJ(1)
C       ENDIF
C       IF (I.EQ. (N+1)) THEN
C           RO=A-ADJ(1)
C       ENDIF
C       DO 150 M=1,NINT(ADJ(2))
C           R=B-DEL(2.0+REAL(M)*DEL)
C           CALL GRNFNC R,RO,NU,D,GRNMT1
C           STTAV M,1=-STT M,9 -STT M-1,9,.,2 0
C           STTAV M,2=-STT M,10 -STT M-1,10,.,2 0
C           DO 140 J=1,4
C               DT 1,J=DT 1,J+ XMT M *GRNMT1 1,J *R DEL*STTAV M,1 +
C                   -GRNMT1 3,J D-NU R*GRNMT1 2,J *R
C                   -DEL*STTAV M,2 *GRNMT1 2,J
C           140 CONTINUE
C       150 CONTINUE
C       IF I.EQ. 1 THEN
C           RO=B-DEL
C       ELSE
C           RO=RO-DEL
C       ENDIF
C   155 CONTINUE
C
C
C*
C*****
C*
C*   AT EACH INCREMENT OF WIDTH DEL ALONG THE PLATE RADIUS, CALCULATE
C*   THE DEFLECTION, SLOPE, MOMENT AND SHEAR, WHICH ARE STORED IN
C*   STT. EACH COLUMN REPRESENTS RO, Y, TH, M, Q, V, W, Y, W, CORRES-
C*   PONDING TO EACH STATION I ACROSS THE RADIUS
C*
C*****
C*
C
C   DO 200 I=1,N+1
C
C       IF I.EQ. 1 THEN

```

```

      RO=B*ADJ(1)
    ENDIF
    IF (I.EQ.(N+1)) THEN
      RO=A*ADJ(1)
    ENDIF
  C
    R=A
    CALL GRNFNC(R,RO,NU,D,GRNMT1)
    R=B
    CALL GRNFNC(R,RO,NU,D,GRNMT2)
    DO 180 J=1,4
      DO 170 K=1,4
        SM(I,J)=A*GRNMT1(K,J)*BCMAT(K)*(-1.0)**K-
          B*GRNMT2(K,J)*BCMAT(K-4)*(-1.0)**K*SM(I,J)
170      CONTINUE
        SM(I,J)=SM(I,J)+DT(I,J)
180      CONTINUE
  C
  C*
  C*****
  C*
  C*      STT(1,1) - RADIUS OF INTEREST
  C*      STT(1,2) - DEFLECTION
  C*      STT(1,3) - DW/DR
  C*      STT(1,4) - RADIAL MOMENT
  C*      STT(1,5) - SHEAR
  C*      STT(1,6) - TANGENTIAL MOMENT
  C*      STT(1,7) - DW^2/DR^2
  C*      STT(1,8) - DW^3/DR^3
  C*      STT(1,9) - RADIAL PLASTIC MOMENT
  C*      STT(1,10) - TANGENTIAL PLASTIC MOMENT
  C*
  C*****
  C*
  C
    STT(1,1)=RO
    STT(1,2)=SM(1,1)/RO
    STT(1,3)=1.0/RO*(SM(1,2)-STT(1,2))
    STT(1,7)=1.0/RO*(SM(1,3)-2.0*STT(1,3))
    STT(1,8)=1.0/RO*(SM(1,4)-3.0*STT(1,7)
    STT(1,4)=-D*(STT(1,7)+NU/RO*STT(1,3))
    STT(1,5)=-D*(STT(1,8)+1.0/RO*STT(1,7)-1.0/RO**2.0*
      STT(1,3))
    STT(1,6)=-D*(1.0/RO*STT(1,3)+NU*STT(1,7))
  C
  C*
  C*      THIS IS A STEP TO PREVENT THE INTERIOR POINTS FROM
  C*      HAVING THE EPSILON THROWN IN.
  C*
  C
    IF I.EQ.1. THEN
      RO=B*DEL
    ELSE
      RO=RO*DEL
    ENDIF

```

```

C
C000 CONTINUE
C
      RETURN
      END

```

```

C
C*
C*.....
C*
C* SUBROUTINE YLD
C*
C* THIS SUBROUTINE TAKES THE PLATE BENDING SOLUTION FOR THE INPUT
C* LOAD INCREMENT AND CALCULATES THE LOAD AT WHICH YIELDING OF THE
C* OUTER FIBER WILL OCCUR. THE LOAD MATRIX IS THEN INCREASED TO
C* THIS VALUE. THIS IS ESSENTIALLY A TIME SAVER, TO AVOID STEPPING
C* UP THE LOAD IN THE ELASTIC RANGE.
C*
C*.....
C*
C* SUBROUTINE YLD KK,IMAX,LDMAT1
C*
C* IMPLICIT REAL*8 A-H,O,Z
C* REAL*8 LDMAT1(50,5),STRMAX,STIMAX
C* INTEGER KK,IMAX,IRMAX,ITMAX
C*
C* INCLUDE 'COMMON.FOR'
C*
C*
C* FIND THE LOCATION ACROSS THE PLATE WHERE THE BENDING MOMENTS
C* ARE THE HIGHEST
C*
C*
C*
C* STRMAX=0.0
C* STIMAX=0.0
C* DO 100 I=1,NINT(ADJ 2 *1)
C*   IF (ABS STI 1,4) .GT. STRMAX THEN
C*     STRMAX=ABS STI 1,4
C*     IRMAX=I
C*   ENDIF
C*   IF (ABS STI 1,4) .GT. STIMAX THEN
C*     STIMAX=ABS STI 1,4
C*     ITMAX=I
C*   ENDIF
C*   IF (STRMAX .GT. STIMAX) THEN
C*     KK=I
C*     IMAX=IRMAX
C*   ENDIF
C* 100 CONTINUE

```

```

      KK=2
      IMAX=ITMAX
      ENDDIF
122    CONTINUE
C
C*
C*   CALCULATE THE EQUIVALENT STRESS AT THE LOCATION WHERE STRESSES
C*   ARE HIGHEST. COMPARE IT TO THE YIELD STRESS. THEN RATIO UP THE
C*   LOAD MATRIX ACCORDINGLY
C*
C
      ERMAX=STT(IMAX,7)*GEOM(5)**2.0
      ETMAX=STT(IMAX,3)*GEOM(5)**2.0/STT(IMAX,1)
      SRMAX=GEOM(6)/GEOM(7)**1.0-GEOM(3)**2.0*ERMAX-GEOM(3)*ETMAX
      STMAX=GEOM(6)/GEOM(7)**1.0-GEOM(3)**2.0*ETMAX-GEOM(3)*ERMAX
      SEMAX=SQRT(SRMAX**2.0-SRMAX*STMAX-STMAX**2.0)
      R=GEOM(6)/SEMAX
      DO 126 J=1,NINT(ADJ(2))
        LDMAT1(J)=R*LDMAT1(J)
126    CONTINUE
      WRITE(*,130) LDMAT1(1)
130    FORMAT(2X,'LDMAT1 = ',F20.10)
C
      RETURN
      END

```

```

C
C*
C*****
C*
C*   SUBROUTINE PLAST
C*
C*   THIS SUBROUTINE CALCULATES PLASTIC MOMENTS USING A TRAPAZOIDAL
C*   RULE INTEGRATION SCHEME. THE SUBROUTINE WILL ONLY HANDLE
C*   LINEAR ELASTIC-PERFECTLY PLASTIC MATERIALS.
C*
C*****
C*
C
      SUBROUTINE PLAST(I
C
      IMPLICIT REAL*8 A-H,O-Z
      REAL*8 MRP,MIP,NL
      INTEGER IZ
      INCLUDE 'COMMON.FOR'

```

```

C
C*
C*****
C*
C*   VARIABLES USED IN THIS SECTION ARE DEFINED AS FOLLOWS
C*
C*   DER - TOTAL RADIAL STRAIN INCREMENT
C*   DET - TOTAL TANGENTIAL STRAIN INCREMENT
C*   DERE - ELASTIC RADIAL STRAIN INCREMENT
C*   DETE - ELASTIC TANGENTIAL STRAIN INCREMENT
C*   DEL - DEPTH INCREMENT
C*   I - RADIAL POSITION STATION NUMBER
C*   IZ - HALF-THICKNESS STATION NUMBER
C*   DS1 - RADIAL STRESS INCREMENT
C*   DS2 - TANGENTIAL STRESS INCREMENT
C*   STRT - MATRIX HOLDING THE TEMPORARY STRESSES AT THE
C*         GIVEN LOADING INCREMENT (1=RADIAL,2=TANGENTIAL)
C*   SET - TEMPORARY EQUIVALENT STRESS - USED FOR YIELD
C*         CRITERIA
C*   SE - EQUIVALENT STRESS FROM PREVIOUS INCREMENT
C*   DEPT - ARRAY CONTAINING THE TEMPORARY STRAIN INCREMENTS
C*         AT THE GIVEN LOADING INCREMENT (1=RADIAL,2=
C*         TANGENTIAL)
C*   R - CONSTANT REFERRING TO THE FRACTION BEYOND YIELDING
C*        THAT OCCURRED DURING THE COURSE OF THE LAST LOAD
C*        INCREMENT
C*   Z - DEPTH
C*
C*****
C*
C
      STT(1,9)=0.0
      STT(1,10)=0.0
      SSUMR=0.0
      SSUMT=0.0
      DEL=GEOM(5)/2.0*(ADJ(6)-1.0)
      Z=0
      E=GEOM(6)/GEOM(7)
      NU=GEOM(3)
C
      DO 500 IZ=NINT(ADJ(6)),1,-1
C
C*
C*****
C*
C*   THIS SECTION CALCULATES THE STRAIN AND STRESS INCREMENTS
C*   CORRESPONDING TO THE GIVEN LOAD INCREMENT. IT THEN ADDS THE
C*   STRESS INCREMENT TO THE PREVIOUS TOTAL STRESS TO CALCULATE THE
C*   EQUIVALENT STRESS. THE EQUIVALENT STRESS IS COMPARED TO THE
C*   UNIAXIAL YIELD STRESS TO SEE IF YIELDING HAS OCCURRED. IF SO,
C*   THE PLASTIC STRAIN INCREMENT IS SET EQUAL TO THE TOTAL STRAIN
C*   INCREMENT
C*
C*****
C*

```



```

C
DER=-STT(1,7)*IZ-10*DEL
DET=-STT(1,3)*IZ-10*DEL/ST(1,1)
DERE=DER-DEPT(1,1Z,1)
DETE=DET-DEPT(1,1Z,2)
DS1=E/(1.0-NU**2.0)*DERE+NU*DETE
DS2=E/(1.0-NU**2.0)*DETE+NU*DERE
STRT(1,1Z,1)=STR(1,1Z,1)+DS1
STRT(1,1Z,2)=STR(1,1Z,2)+DS2
SET(1,1Z)=SQRT(STRT(1,1Z,1)**2.0+STRT(1,1Z,1)*STRT(1,1Z,2)
+STRT(1,1Z,2)**2.0)
C
C*
C* SEE IF YIELD CRITERIA IS EXCEEDED. IF SO, REZERO THE PLASTIC
C* STRAIN INCREMENTS AND THE STRESS INCREMENTS. ALSO, RETURN TO THE
C* MAIN PROGRAM IF THE OUTER FIBER HAS NOT BEGUN TO YIELD
C*
C
IF (SET(1,1Z) .LT. GEOM(6)) THEN
  DEPT(1,1Z,1)=0.0
  DEPT(1,1Z,2)=0.0
  STT(1,9)=0.0
  STT(1,10)=0.0
  IF (IZ .EQ. NINT(ADJ(6))) THEN
    RETURN
  ENDIF
ELSE
  SET(1,1Z)=GEOM(6)
  DEPT(1,1Z,1)=DER
  DEPT(1,1Z,2)=DET
  IF (SET(1,1Z) .LT. GEOM(6)) THEN
    R=(SET(1,1Z)-GEOM(6))/(SET(1,1Z)-SET(1,1Z))
    DEPT(1,1Z,1)=R*DEPT(1,1Z,1)
    DEPT(1,1Z,2)=R*DEPT(1,1Z,2)
    STRT(1,1Z,1)=STR(1,1Z,1)+DS1*(1-R)
    STRT(1,1Z,2)=STR(1,1Z,2)+DS2*(1-R)
  ENDIF
ENDIF
C
Z=Z+DEL
C
500 CONTINUE
C
C*
C*****
C* PERFORM A TRAPAZOIDAL RULE INTEGRATION OF THE STRAIN OVER THE
C* PLATE HALF THICKNESS, AND MULTIPLY BY 2 TO GIVE THE TOTAL RADIAL
C* AND TANGENTIAL PLASTIC MOMENT INCREMENTS STT 1,9 AND STT 1,10
C*
C*****
C
Z=-DEL/2.0
DO 540 IZ=1,NINT ADJ 6 -1

```

```

      Z=Z*DEL
      DMPR=((DEPT(I,IZ,1)-DEPT(I,IZ+1,1))-GEOM(3)*DEPT(I,IZ,2)
      +DEPT(I,IZ+1,2))*Z/2.0
      DMPT=((DEPT(I,IZ,2)+DEPT(I,IZ+1,2))-GEOM(3)*DEPT(I,IZ,1)
      +DEPT(I,IZ+1,1))*Z/2.0
      SSUMR=SSUMR-DMPR
      SSUMT=SSUMT+DMPT
540  CONTINUE
C
      STT(I,9)=2.0*GEOM(6)/GEOM(7)/(1-GEOM(3)**2.0)*DEL*SSUMR
      STT(I,10)=2.0*GEOM(6)/GEOM(7)/(1-GEOM(3)**2.0)*DEL*SSUMT

      WRITE(*,600) I
600  FORMAT(2X,'LAT POS = ',I2)
      WRITE(*,700) STT(I,9),STT(I,10)
700  FORMAT(2X,'MRP = ',F20.12,' MTP = ',F20.12)
C
      RETURN
      END

```

```

C
C*
C*****
C*
C*  SUBROUTINE UPDATE
C*
C*  THIS SUBROUTINE TAKES THE INCREMENTAL STRESSES, STRAINS,
C*  DEFLECTIONS, MOMENTS, ETC AND ADDS THEM TO THE PREVIOUS TOTALS
C*  ONCE CONVERGENCE OF PLASTIC MOMENTS HAS OCCURRED.
C*
C*****
C*
C  SUBROUTINE UPDATE(STT1,IMAX,OUT,LDMAT1,LL,LDMAT0)
C
C  IMPLICIT REAL*8 (A-H,O-Z)
C  REAL*8 STT1(50,2),LDMAT0(50),LDMAT1(50,NU,DEL,E)
C  INTEGER NN,OUT,LL
C  $INCLUDE 'COMMON.FOR'
C
C  NN=NINT(ADJ/2)+1
C  E=GEOM(6)/GEOM(7)
C  DEL=GEOM(5)/2.0/ADJ(6)+1.0
C  NU=GEOM(3)
C
C  DO 1000 I=1,NN
C    DO 905 J=LL,10
C      ST(I,J)=ST(I,J)+STT(I,J)
905  CONTINUE

```

```

      LDMATO(I)=LDMAT(I)+LDMATO,I)
C
      DO 980 IZ=1,NINT(ADJ*G)
C#
C#      IF THE YIELD CRITERIA HAS BEEN EXCEEDED, THEN ADD IN THE
C# PLASTIC STRAIN INCREMENTS AND UPDATE THE TOTAL STRESS TO
C# EQUAL THE TEMPORARY STRESS CALCULATED IN THE SUBROUTINE
C# PLAST
C#
C      IF (SE(I,IZ).GT. GEOM*6) THEN
          EP(I,IZ,1)=EP(I,IZ,1)+DEPT(I,IZ,1)
          EP(I,IZ,2)=EP(I,IZ,2)+DEPT(I,IZ,2)
          STR(I,IZ,1)=STR(I,IZ,1)
          STR(I,IZ,2)=STR(I,IZ,2)
          Z=(GEOM*5)/2.0/(ADJ*6-1.0)**IZ-1
      ELSE
C#
C#      IF THE YIELD CRITERIA HAS NOT BEEN EXCEEDED, THEN UPDATE THE
C# STRESSES AND EQUIVALENT STRESSES USING THE ELASTIC EQUATIONS
C#
C      STR(I,IZ,1)=-E/(1.0-NU**2.0)*(ST(I,7)**IZ-1)*DEL
          - EP(I,IZ,1)+NU*ST(I,3)**IZ-1)*DEL*ST(I,1)
          STR(I,IZ,2)=-E/(1.0-NU**2.0)*(ST(I,3)**IZ-1)*DEL*ST(I,1)
          - EP(I,IZ,2)+NU*ST(I,7)**IZ-1)*DEL
          SE(I,IZ)=SQRT(STR(I,IZ,1)**2.0+STR(I,IZ,2)**2)
          - STR(I,IZ,2)+STR(I,IZ,2)**2
      ENDIF
          IF (ABS(EP(I,IZ,1)).GT. GEOM*9) .OR. ABS(EP(I,IZ,2)
          - .GT. GEOM*9) THEN
              OUT=1
          ENDIF
980      CONTINUE
1000 CONTINUE
C
      LL=2
      RETURN
      END

```



```

      WRITE(15,165)
      WRITE(*,165)
      FORMAT(/,2X,' DEPTH ',2X,' ERP ',2X,' SRAD ',
      2X,' ETP ',2X,' STAN ',2X,' SE ',/)
      DO 200 IZ=1,NINT(ADJ(6))
      IF (SE(1,IZ) .GT. GROM(6)) THEN
      Z=(GROM(5)/2.0/(ADJ(6)-1.0))*(IZ-1)
      WRITE(15,180) Z,EP(1,IZ,1),STR(1,IZ,1),EP(1,IZ,2),
      STR(1,IZ,2),SE(1,IZ)
      WRITE(*,180) Z,EP(1,IZ,1),STR(1,IZ,1),EP(1,IZ,2),
      STR(1,IZ,2),SE(1,IZ)
      FORMAT(2X,F6.4,3X,F10.8,3X,F8.1,3X,F10.8,3X,F8.1,
      3X,F8.1)
      ENDIF
200    CONTINUE
      ENDIF
1000  CONTINUE
      ENDIF
      RETURN
      END

```

```

COMMON GEOM(9),ADJ(7),ST(50,10)
COMMON /PLSCM/ EP(50,50,2),STT(50,10),DEPT(50,50,2),STR(50,50,2)
SE(50,50),STRT(50,50,2),SET(50,50)
REAL*8 ADJ,ST,EP,STT,DEPT,STR,SE,STRT,SET

```

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